Check for updates

RESEARCH ARTICLE

REVISED Enumeration Class of Polyominoes Inscribed in James Abacus and Related ECO

[version 3; peer review: 2 approved, 1 approved with reservations, 1 not approved]

Eman F. Mohommed, jalal Abd Jassim

mathematic, Mustansiriyah University Department of Mathematics, Baghdad, Baghdad Governorate, Iraq

v3 **First published:** 06 Jan 2026, **15**:11
https://doi.org/10.12688/f1000research.172910.1
Second version: 15 May 2026, **15**:11
https://doi.org/10.12688/f1000research.172910.2
Latest published: 30 Jul 2026, **15**:11
https://doi.org/10.12688/f1000research.172910.3

Abstract

This paper studies a class of polyominoes. The new class is defined through a representation on the James abacus through nested chain which denoted Ω -nested Abacus. Depended on partition, beta number, nested chain we defined the structural condition this class. A local transformations, called SSPT-transformation and MSPT-transformation are formulated on the even chains. Then, we drive explicit formulas for the number of position in each chain, number of even chain and the total number of all chains. The structure of new class lead to enumeration formulas for the objects produced after application new transformation. To enhance further, based on these classes, generating functions are also being formulated by employing enumeration of combinatorial objects (ECO). In ECO method, each object is obtained from smaller object by making some local expansions. These local expansions are described in a simple way by a succession rule which can be translated into a function equation for the generating function.

Keywords

Polyominoes, Class, Enumeration, Succession-rule, Abacus diagram

This article is included in the Fallujah Multidisciplinary Science and Innovation gateway.

Open Peer Review

Approval Status ✗ ✓ ? ✓

	1	2	3	4
version 3 (revision) 30 Jul 2026				
version 2 (revision) 15 May 2026		✓ view	? view	✓ view
version 1 06 Jan 2026	✗ view	? view		

1. **Ayesha Qureshi**, Sabanci Universitesi (Ringgold ID: 52991), Istanbul, Turkey
2. **Hayder R. Hashim**, University of Kufa, Kufa, Iraq
3. **Ali A. Aubad**, University of Baghdad, Baghdad, Iraq
4. **Alina Alb Lupas**, University of Oradea, Oradea, Romania

Any reports and responses or comments on the article can be found at the end of the article.

Corresponding author: Eman F. Mohommed (emanfatel@uomustansiriyah.edu.iq)

Author roles: Mohommed EF: Methodology, Supervision, Writing – Original Draft Preparation, Writing – Review & Editing; **Abd Jassim j**: Funding Acquisition, Software, Writing – Review & Editing

Competing interests: No competing interests were disclosed.

Grant information: The author(s) declared that no grants were involved in supporting this work.

Copyright: © 2026 Mohommed EF and Abd Jassim j. This is an open access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

How to cite this article: Mohommed EF and Abd Jassim j. **Enumeration Class of Polyominoes Inscribed in James Abacus and Related ECO [version 3; peer review: 2 approved, 1 approved with reservations, 1 not approved]** F1000Research 2026, **15**:11 https://doi.org/10.12688/f1000research.172910.3

First published: 06 Jan 2026, **15**:11 https://doi.org/10.12688/f1000research.172910.1

REVISED **Amendments from Version 2**

This version has been substantially revised in response to the reviewers' comments. The definitions of the Nested Chain Abacus and the Ω-nested Abacus have been clarified and reformulated to improve mathematical precision and consistency. The structural assumptions of the proposed model have been explicitly stated, and additional remarks have been included to explain the role of the unique central position. The proofs of Remark 10, Lemma 11, Theorem 16, and Theorem 18 have been expanded by providing the missing mathematical arguments and intermediate steps. The descriptions and captions of Figures 4, 5, and 8 have been revised to improve clarity and better illustrate the proposed constructions. Furthermore, the section on generating functions has been significantly extended by providing a detailed description of the ECO method, the generating tree, the succession rule, the recurrence relation, and the derivation of the generating function. The manuscript has also undergone a thorough revision to improve notation, mathematical consistency, and the overall presentation.

Any further responses from the reviewers can be found at the end of the article

Introduction

Polyominoes are finite configurations composed of unit squares, known as ominoes, that are joined edge to edge to form a connected interior, as illustrated in Figure 1.

An *n*-polyomino (a polyomino consisting of *n* ominoes) is defined up to translation, and the concept is commonly attributed to Golomb.[1] In contemporary research, *n*-polyominoes have attracted significant attention from computer scientists, physicists, mathematicians, and biologists. Despite extensive studies, enumerating n-polyominoes remains a challenging and unresolved problem in combinatorial geometry, representing one of its most fundamental open questions.[2–5] There is no closed-form equation for *n*-polyominoes, and the problem has been solved up to $n < 56$.[3] No closed-form expression for the enumeration of *n*-polyominoes is currently known. Due to the complexity of this problem, several simpler subclasses of *n*-polyominoes have been formulated and extensively examined in the literature.[6–8] E.F. provide a new representation of *n*-polyominoes (Plyominoes with *n* ominoe) using one of the graphical representations of partition (James-diagram) called nested chain abacus (N.C.A.) with *b*-columns and *d*-rows.[9] The Nested Chain Abacus (N.C.A.) offers a novel framework for representing any *n* connected omino, including empty ones (holes), through the use of a beta number. In this new representation, *n*-polyominoes are organised into a series of nested chains.[10] A Nested Chain Abacus (N.C.A.) is an *n*-polyomino inscribed within a James diagram, consisting of both outer and inner chains. The inner chains are numbered from 1 to *n*, where *n* is a positive integer, and chain 1 is the innermost chain. No intersections occur among any of the chains. Through this new representation, and for the first time, each *n*-polyomino has been systematically associated with a unique code. Next Figure 2 illustrates an example of an N.C.A. with six columns, five rows, and three chains.

Although the nested chain abacus provides a general representation pf polyominoes using the James Abacus provides a general representation of polyominoes using the James Abacus, its general structure is not specifically designed for systematic combination enumeration through local transformation. In particular, the absence of structural restriction on

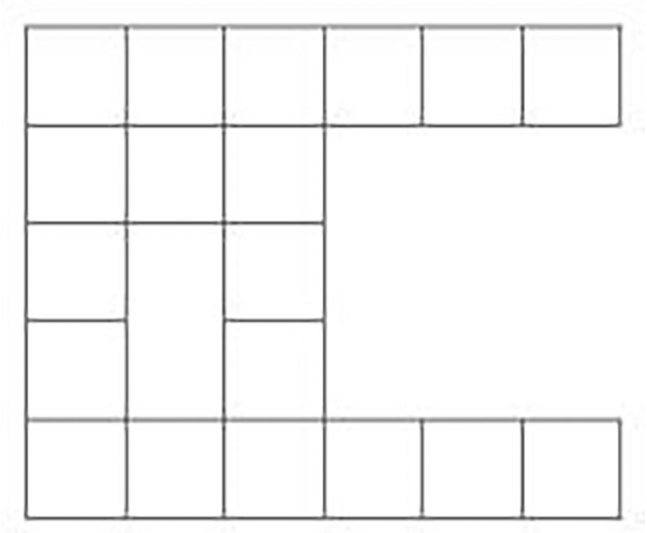


Figure 1. 22-polyominoes (polyominoes with 22 connected ominoes).

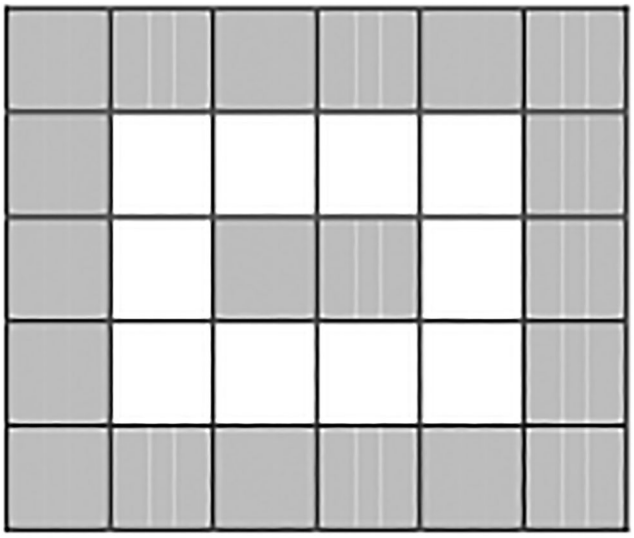


Figure 2. Nested chain abacus with 6 columns, 5 rows and 3 chains.

the arrangement of chains makes it difficult to derive explicit counting formulas and generating function. This limitation properties required for transformation-based enumeration.

To address this issue, we introduce the Ω-nested Abacus, a restricted family of Nested chain Abacus representations characterized by specific conditions imposed on the odd and even chains. These conditions enable the definition of local transformation and provide a convenient framework for deriving explicit enumeration formulas using ECO methodology.

Terminologies and definition

This section introduced some importuned definitions:

Definition 1. [13] A partition of a positive integer, k, is a sequence of integers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\sum_{i=1}^{n} \lambda_i = k$.

Example 1. $\lambda = (6,6,3,2,1,1)$ is a partition of 19

Definition 2. [13] Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition of a positive integer (k) where (n) denotes the number of parts of the partition. The corresponding beta number sequence is defined by $\beta_i = \lambda_i + n - i$ where $1 \leq i \leq n$. The sequence $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is called the beta-number sequence.

Example 2. [14] Consider of Example 1, $\lambda = (6,6,3,2,1,1)$ then beta number of λ is

$$\beta_i = \lambda_i + n - i, \quad \text{where } i = 1,2,3,4,5,6 \text{ and } n = 6$$

And hence

$$\beta_1 = 11, \beta_2 = 10, \beta_3 = 6, \beta_4 = 4, \beta_5 = 2, \beta_6 = 1$$

Thus beta number sequence is

$$\{11,10,6,4,2,1\}.$$

Definition 3. [14] Let $b \geq 2$ denote the number of columns (runners) , and let λ be a partition with beta number sequence $\{\beta_1, \beta_2, \ldots, \beta_n\}$. The James Abacus with b columns is a graphical representation of λ obtained by arranging the non negative integers 0,1, … from left to right and row by row.

A bead is placed at each position whose label belongs to the beta- number sequence of, all remaining position are left vacant.

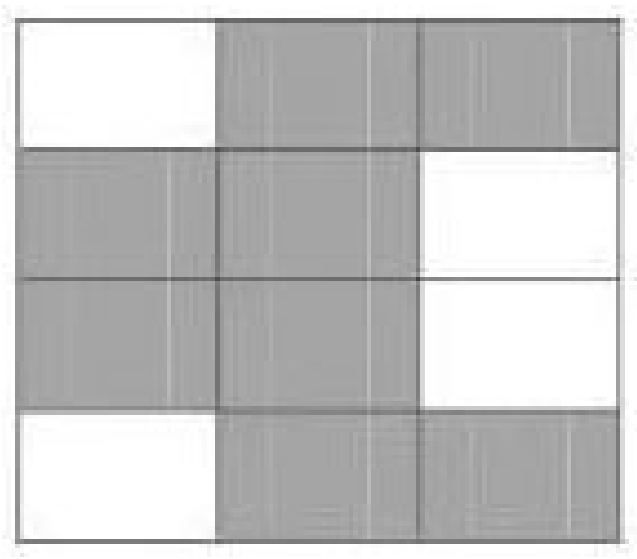


Figure 3. James's Abacus with 3 columns, 4 rows and 6 beta.

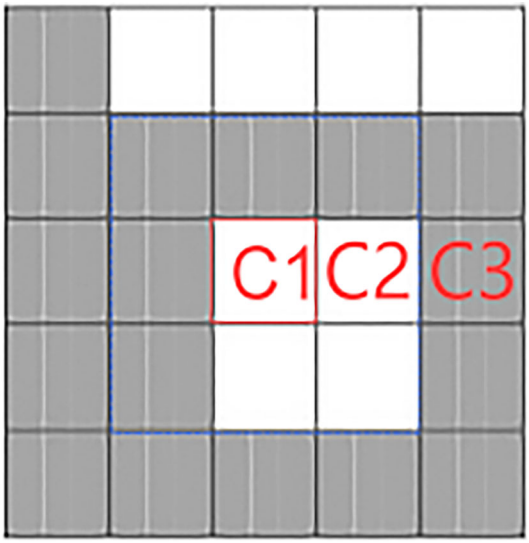


Figure 4. N.C.A. consisting of 5-columns and 5-rows decomposed into three nested chain (C_1, C_2, C_3) representing the partition of 94. The corresponding beta-number sequence is {0,5,6,7,8,9,10,11,14,15,16,19,20,21,24}. Grey cells denoted beta position whereas white cells denote the Vance positions.

Consider Example 1, the James Abacus with 1,2,3,4,6,7,10,11 beta position as show in next Figure 3.

Definintion 4. [9] The nested chain Abaus (N.C.A) treads the abacus A as the two dimensional grid of d rows and b columns. Where this N.C.A structure is utilized to represent various polyominoes configurations.

Each position on the abacus is determined by its coordinates (m,j), where $1 \leq m \leq d$, represent the row index, and $1 \leq j \leq b$ represent the column index. The beta-number β corresponding to the coordinate (m,j) is given by the mapping $\beta = (m-1)b + (j-1)$.

The N.C.A structurally partitioned into finite sequence of concentric, non-over lapping, and clouded boundary path called chain denoted by $C_1, C_2, \ldots, C_r$ where r is the number of chains such that $\bigcap_{i=1}^{r} C_i = \varnothing$ and $\bigcup_{i=1}^{r} C_r = A$.

Not. For the formal construction, geometric properties $\bigcap_{r=1}^{\varepsilon} C_r$s and coding algorithm of the N.C.A the reader referred to Reference 9.

Next Figure 4 gives an example of N.C.A. with 3 chains represented of 94, where the beta number sequence is {0,5,6,7,8,9,10,11,14,15,16,19,20,21,24}.

Definition 5. A chain C_r is said to be connected chain if every position (m, j) in C_r is a beta-position. Equivalenty contain no empty position.

Next Figure 5 gives an example of N.C.A. represented by a beta sequence {0,1,2,3,4,5,7,8,9,10,12,14, 15,19,20,21,22,23,24} with one connected chain (chain 1).

Definition 6. Two beta numbers β_i and β_j represented on Nested Chain Abacus with b-columns and d-rows are said to be connected iff either of the following condition is satisfied.

1- $|\beta_i - \beta_j| = 1$ provided that β_i, β_j are Located in the same row.

2- $|\beta_i - \beta_j| = b$ provided that β_i, β_j are Located in the same column.

For example, the beta set

$$\{0,5,6,7,8,9,10,11,14,15,16,19,20,21,24\}.$$

From Figure 4 is a connected.

Definition 7. Each nested chain abacus (N.C.A.) with b- columns and d- rows is called a polyamines if every two beta numbers are connected.

Definition 8. Onsider a Nested Chain Abacus(A) consisting of b-columns and d-rows, with chains $C_1, C_2, \ldots, C_r$ orders from innermost chain to the outermost chain where (b=d) and b is odd. The N.C.A is called Ω-nested Abacus if it satisfies the following structural condition.

1. If the unique position in C1 is occupied by a bead (beta position), then the second chain C2 contains at last five beta positions.

2. If the unique position in C1 is vacant, then the second chain C2 contains at least one beta position.

3. Every chain with an even index may contain both beta position and vacant position.

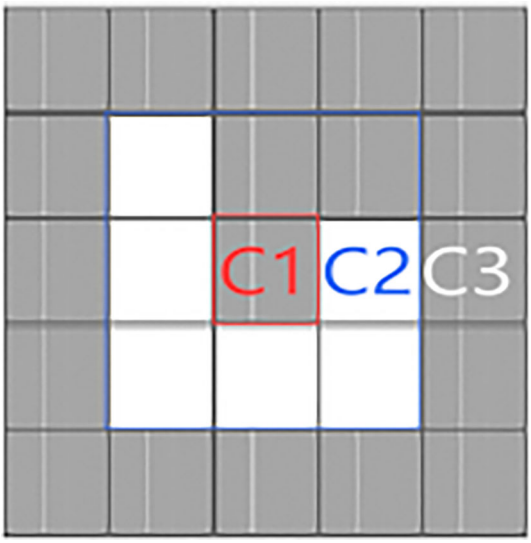


Figure 5. Example of N.C.A illustrating a connected third chain as defined in Definition 5.

4. Every chain with an odd index greater than one forms a connected chain.

Any polyomino represented by Ω-nested Abacus is called Ω-nested Abacus polyomino.

Novelty of the Ω-nested Abacus

The proposed Ω-nested Abacus should e regarded as a restricted subclass of the N.C.A, rather than as a replacement for the original representation. While th N.C.A provide a general framework for representing arbitrary polyomines on James Abacus, it dose not impose structural constraints that facilitate systematic enumeration through local transformation.

The Ω-nested Abacus intrduses explicit condition on the parity and connectivity of the chains. In particular, every odd chain is required to remain connected, whereas every even chain is permitted to contain both occupied and empty beta positions.

Remark 9.

1- For a N.C.A with (b=d) and b is odd, the innermost chain C1 consists of a unique central position (see Reference 9).

2- The additional structural in Definition 8 distinguish the Ω -nested Abacus form the general N.C.A. These condition are imposed to ensure that the corresponding polyomino remain connected under the SSPT and MSPT transformation.

Figure 6 gives an example of Ω-nested Abacus with four chains, where

Not Every polyamines inscribed in a nested chain-abacus is Ω-nested Abacus.

Figure 4 given an example of N.C.A. but not Ω-nested Abacus, while Figure 6 given an example of Ω-nested Abacus.

Not. Throughout our result the underlying new family (Ω-nested Abacus) assumed $b = d$ where b is odd number. Since every odd chain is completely filled, so only even chain well be contribute to the generating process.

The algorithm construct to generate classes of new family depended on the following transformation.

Definition 9. Single – beta transformation (SPT)

Let an Ω-nested Abacus with b column, d rows and C_i chains. Suppose that a beta number $\beta = (m-1)b + (j-1)$ in chain C_i is located in row m and column j such that $1 \le m \le d$, $1 \le j \le b$ and $1 \le i \le r$. A single beta transformation (SPT) in chain C_i is local transformation ($\beta \rightarrow \grave{\beta}$) within the same chain where

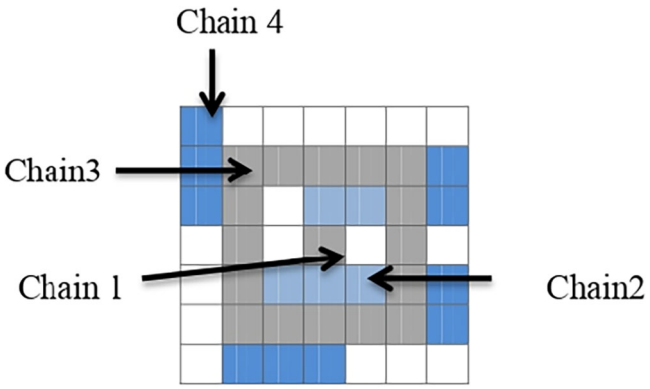


Figure 6. Ω-nested Abacus with four chains.

$$\grave{\beta} = \begin{cases} (m-1)b+(j-2) & if \quad i+1 \le m \le (r-i+1),\ j = b-i+1 \\ mb+(j-1) & if \quad i \le m \le (r-i),\ j = i \\ (m-1)b+j & if \quad m = i,\ i+1 < j \le b-i+1 \\ (m-2)b+(j-1) & if \quad m = d-i+1,\ i \le j < b-i. \end{cases}$$

Lemma 10. The maximal number of Single Beta transformations (SPT) that can be performed in a chain C_i $(i>1)$ is equal to the number of positions in C_i.

Proof. Let C_i be a chain consisting of N_i positions. By Definition 9, a single Beta transformation acts locally on beta position and moves it to the next position within the same chain according to the prescribed transformation rule. Since every position in C_i has a unique successor and every position can be selected exactly once as the starting point of a SSpt, each position determines one distinct SSPT. Therefore, the total number of distinct SSPT is equal to the number of position in C_i, namely N_i, consequently, after N_i successive SSPT, every beta position returns to its original location, completing on full cycle of the chain. Hence, the maximal number of SSPT in C_i is equal to the number of position in that chain.

Let C_i (i>1) be a chain consisting of N_i

Figure 7 illustrates Single – beta transformation (SPT)

Not.

1- SSpt is a SPT transformation in one chain.

2- MSpt is a SPT transformation in all chains.

Enumeration of the Ω-nested Abacus class

Next, the enumeration of Ω-nested Abacus with b columns and d rows is presented.

Lemma 11. Let A be Ω-nested Abacus with b column, d rows and $C_1, C_2, \ldots, C_n$ then, the number of position in any chains C_i is $|C_i| = 8(n-1)$.

Proof. Based on Remark 8 where d=b is odd, the innermost chain C_1 colonists of a unique position. According to Definition 4, every chain C_n (n>1) is the closed boundary of the remaining sub-grid after removing the inner chain $C_1, C_2, \ldots, C_{n-1}$. The second chain (C_2) is formed by the eight boundary position surround the unique central position. Then

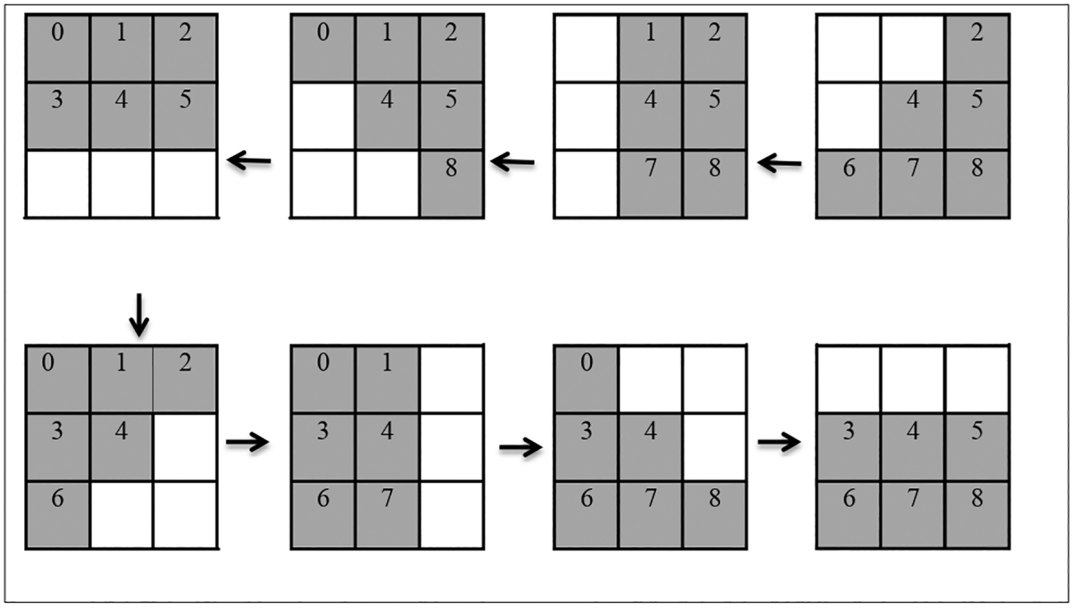


Figure 7. Illustrates Single – beta transformation (SPT) on Ω-nested Abacus.

$$|C_2| = 4+4 = 8$$

$$|C_3| = |C_2| + 8 = 8.2 = 16$$

$$|C_4| = |C_3| + 8 = 8.3 = 24 \ldots \ldots \ldots |C_n| = |C_{n-1}| + 8 = 8(n-1)$$

Corollary 12. Based on Remark 10 and Lemma 11, the number of (SSPT) transformation in chain C_n is equal to $8(n-1)$.

Lemma 13. Let Ω-nested Abacus be a polyomino intercept in N.C.A. with b columns and d rows such that $b = d$ and b is odd. Then the total number of chain is $\frac{b+1}{2}$.

Proof. Based on Ω-nested Abacus structure is symmetric with respect to a central column (central chain C_n) then for all i-th chains, the boundary columns must satisfy $i \leq b - i + 1$, thus $2i \leq b - 1 \ \rightarrow \ i \leq \frac{b+1}{2}$. Thus the number of even chain is $\frac{b+1}{2}$ chains.

Proposition 14. Let Ω-nested Abacus be a polyomino intercept in NCA with b columns and d rows such that $b = d$ and b is odd. Then the number of even chain is $\frac{b-1}{4}$.

Provided that $b \equiv 1 \ (mod\, 4)$.

Proof. Based on Lemma 13 the number of chain in Ω-nested Abacus is $t = \frac{b+1}{2}$. The even chains among $1, 2, \ldots, t$ are $2, 4, \ldots, \lfloor \frac{t}{2} \rfloor$. Since $t = \frac{b+1}{2}$ is even then $\frac{t}{2} = \frac{\frac{b+1}{2}}{2}$ is even since the first chain is odd and fixed. Then there are Since $t = \frac{b+1}{2}$ is even then $\frac{t}{2} = \frac{\frac{b+1}{2}-1}{2} = \frac{b-1}{4}$, then the number of even chain is $\frac{b-1}{4}$.

Next we found the number of Ω-nested Abacus if we apply SSPT-Transformation in chain i.

Lemma 15: Let Ω-nested Abacus be a polyomino intercept in NCA with b columns and d rows such that $b = d$, b is odd and let n be even number. Then the number of Ω-nested Abacus generating by employ SSPT-Transformation exactly chain n is $8(n-1)$.

Proof. By Lemma 11 the number of admissible positions in chains n is $8(n-1)$. Since move each beta position yields a distinct Ω-nested Abacus under SSPT-transformation obtain in chain n with $8(n-1)$ position is $8(n-1)$.

Theorem 16: Let Ω-nested Abacus be a polyomino intercept in N.C.A. with b columns and d rows such that $b = d$, b is odd and let n be even number. Then the number of Ω-nested Abacus generating by employ SSPT-Transformation is

$$\sum_{\forall n} \binom{R}{(n-1)8}$$

Where $R = \sum_{n}^{\lfloor \frac{b+1}{4} \rfloor} (n-1)8$, $b \equiv 1 \ (mod\, 4)$

Proof. Based on Lemma 10, the number of positions in chain n is (n−1)23 by Lemma 11 and Definition 8, there are b−14 chain with beta and empty beta position in Ω-nested Abacus. The maximal number of transformations in chain n is equal to ((n−1)8). Since each move generates a new Abacus (polyominoes), thus there are ((n−1)8) of Ω-nested Abacus generated by employing SSPt transformation. As a result of this, there are

$$\sum_{\forall n} \binom{R}{(n-1)8}$$

Ω-nested Abacus.

Theorem 17. Let Ω-nested Abacus be a polyomino intercept in N.C.A. with b columns and d rows such that $b = d$, b is odd and let n be even number. Then the number of Ω-nested Abacus generating by employ SSPT-Transformation exactly one even chain.

$$\sum_{n=2}^{\lfloor \frac{b+1}{4} \rfloor} 8(n-1).$$

Where $b \equiv 1 \ (mod\, 4)$

Proof. Based on Lemma 11 the number of positions in chain n is $8(n-1)$, any position in the chain will be determines one distinct SSPT in that chain. Thus all even chain $C_2, C_4, \ldots,$ gives $\sum_{n=2}^{\lfloor\frac{b+1}{4}\rfloor} 8(n-1)$.

Theorem 18. Let Ω-nested Abacus be a polyomino intercept in N.C.A. with b columns and d rows such that $b = d$, b is odd and let n be even number. Then the number of Ω-nested Abacus generating by employ MSPT-Transformation exactly chain n is

$$\prod_{\forall n} 8(n-1)$$

Where $b \equiv 1 \ (mod\, 4)$ and $n = 2,4, \ldots.., \frac{b=1}{4}$

Proof. Based on Lemma 11, the number of admissible beta and empty positions in chain Cn is $8(n-1)$. Since all odd chains are connected and remain fixed, they do not participate in the transformation. By Lemma 15 the number of Ω-nested Abacus generating by employ SSPT-Transformation applying an SSPT to an even chain produces exactly 8 $(n-1)$ distinct Ω-nested Abacus. Furthermore, by Definition 4, the chains of a nested chain Abacus are pairwise disjoint. Hence, each SSPT acts locally within its own even chain and dose not affect the position of any other chain. Therefore, the transformations performed on different even chains are mutually independent and can be applied simultaneously. Consequently, an MSPT is obtained by applying an SSPT to every even chain at the same time. Since each even chain contributes independently 8(n-1) admissible transformations, the total number of Ω-nested Abacus generated after application SSPT-Transformation

$$\prod_{2\leq n\leq\frac{b-1}{2}, \text{n even}} 8(\text{n}-1)$$

Generating function with respect to chains

In this section, the method described in[11,12] is employed to enumerate the Ω-nested Abacus class, representing polyominoes inscribed within a James diagram. The ECO (Enumerating Combinatorial Objects) method has previously been applied to the enumeration of various polyomino classes.[12] This approach is based on a *succession rule.*

Generating Function (G.F.)

Let C_{2k} be even chain of Ω-nested Abacus, based on Lemma 11 the number of positions in any chain is

$$M_k = \lceil C_{2k} \rceil = 8(n-1).$$

We apply SSPT-Transformation inside the chains such that no previously select position may be chose again. Thus, in initial stage there are M_k choices (M_k of Ω-nested Abacus), after one insertion we have only $M_k - 1$ position ($M_k - 1$ of Ω-nested Abacus), after two distinct insertion there are only $M_k - 2$ position ($M_k - 2$ of Ω-nested Abacus as shown in Figure 8), and so on. Thus, there are

$$a_n^k = M_k(M_k-1)(M_k-2)\ldots(M_k-n+1), 2 \leq n \leq M_k$$

Above process is encoded by a generating tree. The root labeled by M_k, each node produces j children, every one of j children product $j-1$. Thus the local succession rule is

$$\vartheta = \begin{cases} M_k \\ j \quad \rightarrow \quad (j-1)^j . 1 \leq j \leq M_k \end{cases}$$

Assume that a_{n+1}^k denoted the number of nodes at level n ϑ yield the recurrence

$$a_{n+1}^k = (M_k - n)a_n^k$$

Where $a_0^k = 1, 2 \leq n \leq M_k$.

$$a_n^k = \frac{M_k!}{(M_k-n)!}.$$

Using the ordinary level-generating polynomial of the even chain

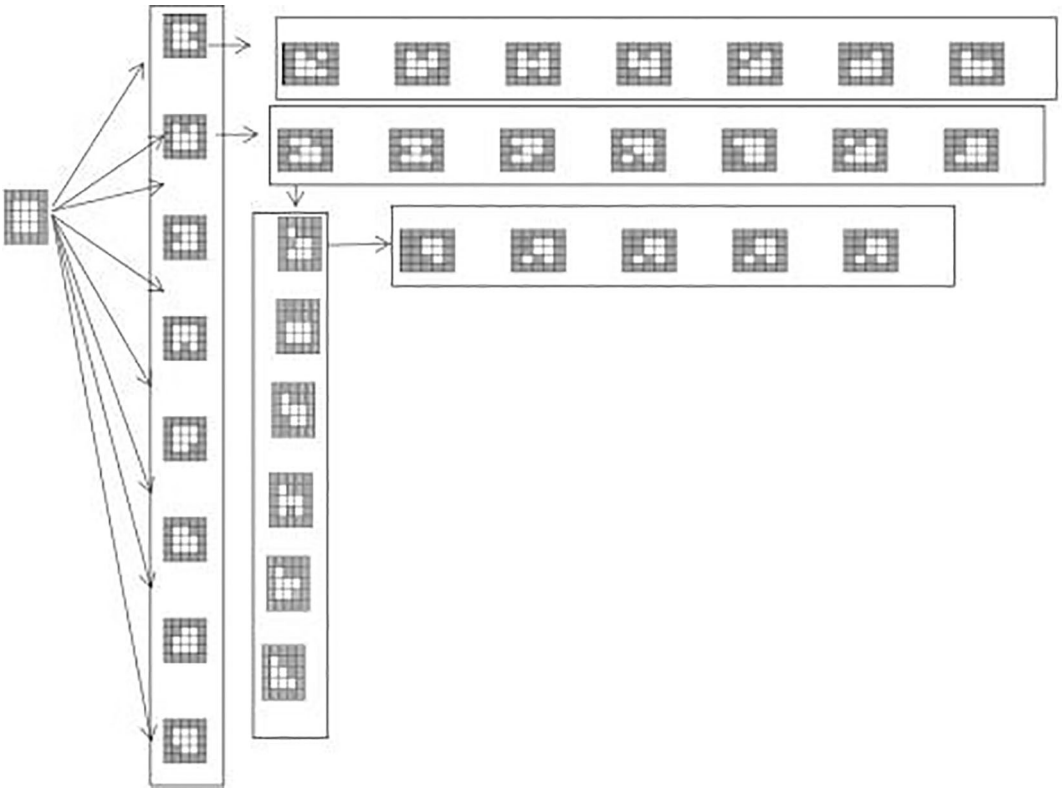


Figure 8. First and second level of ϑ using Ω-nested Abacus class with 5 columns, 5 rows and 3 chain.

$$C_k(x) = \sum_{n=2}^{M_k} (M_k)_n x^n$$

$$C_k(x) = \sum_{n=2}^{M_k} \frac{(M_k)_n}{n!} x^n$$

Since

$$\frac{(M_k)_n}{n!} = \frac{M_k!}{(M_k - n)!n!} = \binom{M_k}{n}$$

then,

$$C_k(x) = \sum_{n=2}^{M_k} \binom{M_k}{n} x^n = (1+x)^{M_k}$$

$$C_k(x) = \sum_{n=2}^{M_k} \binom{M_k}{n} x^n = (1+x)^{8(2k-1)}$$

Then the generating function

$$\mathrm{C_k(z,x)} = \sum_{\mathrm{k=1}}^{\mathrm{m}} \mathrm{Z^k (1+x)^{16k-8}}$$

$$C_k(z,x) = (1+x)^{-8} \sum_{k=1}^{m} Z^k (1+x)^{16k}$$

$$C_k(z,x) = (1+x)^{-8} \sum_{k=1}^{m} \left(Z(1+x)^{16}\right)^k$$

Let $r = Z(1+x)^{16}$

$$C_k(z,x) = (1+x)^{-8} \sum_{k=1}^{m} r^k$$

using the closed form sum for a finite geometric progression

$$\sum_{k=1}^{m} r^k = \frac{r(1-r^m)}{1-r}$$

Then

$$C_k(z,x) = (1+x)^{-8} \frac{r(1-r^m)}{1-r}$$

$$C_k(z,x) = (1+x)^{-8} \frac{Z(1+x)^{16}\left(1-\left(Z(1+x)^{16}\right)^m\right)}{1-Z(1+x)^{16}}$$

$C_k(z,x)$ is generating function to enumerated the number of Ω-nested Abacus.

Conclusion

This study examines a class of polyominoes known as the Ω-nested Abacus. Initially, we provide a characterization of a new class based on specific geometric constraints defined by rows, columns, and chains. A series of operations on polyominoes is then introduced using a partition-theoretic construct known as the beta number. Furthermore, a set of operations on polyominoes is developed using a partition-theoretic construct known as the beta number, allowing for localized transformation of the structure. Furthermore, A succession rule is formulated based on generating tree techniques to describe the growth of these object. Based on this framework a recursive method is established for the systematic generation of Ω-nested Abacus configuration of a given size through the use of generating trees.

Ethical approval

Ethical approval was not required for this study.

Data availability statement

No data availability with this atrial as the study is purely theoretical and does not involve the generating analysis or use of any datasets.

References

1. Golomb S: **Checkerboards and polyominoes.** *Am. Math. Mon.* 1954; **61**(10): 675682.
Publisher Full Text
2. Flórez R, José Ramírez L, Villamizar D: **Restricted bargraphs and unimodal compositions.** *Journal of Combinatorial Theory, Series A.* 2024; **208**: 105934.
Publisher Full Text
3. Arabi A, Belbachir H, Dubernard JP: **Enumeration of plateau polycubes with respect to their lateral area.** *Indian Journal of Pure and Applied Mathematics.* 2024; **55**: 538–554.
Publisher Full Text
4. Jensen C: **Counting polyominoes: a parallel implementation for cluster Computing.** *Proceedings of the 2003 International Conference on Computational Science: Part III, ICCS'03.* Springer-Verlag; 2003; pp. 203–212.
5. Castiglione G, Restivo A: **Ordering and convex polyominoes.** *The International Conference on Machines, Computations, and Universality.* Berlin, Heidelberg: Springer Berlin Heidelberg; September 2004; pp. 128–139.
6. Del Lungo A: **Polyominoes defined by two vectors.** *Theor. Comput. Sci.* 1994; **127**(1): 187–198.
Publisher Full Text

7. Castiglione G, Frosini A, Restivo A, *et al.*: **A tomographical characterisation of convex polyominoes.** *Poitiers, Proc. of Discrete Geometry for Computer Imagery 12th International Conference, DGCI 2005, in: Lecture Notes in Computer.* 2005; pp. 115–125.

8. Castiglione G, Restivo A: **Reconstruction of L-convex polyominoes.** *Electron. Notes Discret. Math.* 2003; **12**: 290–301.
Publisher Full Text

9. Mohommed EF: **Topological Structure of Nested Chain Abacus.** *Iraqi Journal of Science.* 2020; 153–160.
Publisher Full Text

10. Mohommed EF: **Constructing a Nested Chain in James Abacus Diagram.** *J. Phys. Conf. Ser.* 2019; **1294**(3): 032019. IOP Publishing.
Publisher Full Text

11. Duchi E, Fedou J-M, Rinaldi S: **From object grammars to ECO systems.** *Theoretical computer science.* 2004; **314**(1-2): 57–95.
Publisher Full Text

12. Banderier C, Wallner M: **Lattice paths with catastrophes.** *Discrete Mathematics & Theoretical Computer Science.* 2017; **19**(1). Analysis of Algorithms.
Publisher Full Text

13. Andrews G: *The theory of partition (No. 2)*. Cambridge University Press; 1998.

14. James G: **Some combinatorial result involving young diagrams.** *In Mathematical proceeding of the cambridge philosophical society.* 1978; **83**: 1–10.
Publisher Full Text

Open Peer Review

Current Peer Review Status: ✗ ✓ ? ✓

Version 2

Reviewer Report 16 July 2026

https://doi.org/10.5256/f1000research.197917.r497664

© **2026 Lupas A.** This is an open access peer review report distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

✓ **Alina Alb Lupas**
[1] University of Oradea, Oradea, Romania
[2] University of Oradea, Oradea, Romania

The paper introduces and studies a new class of polyominoes defined by unique representations on the James abacus using nested chains, termed the **Ω-nested Abacus**. It establishes structural conditions for this class and develops local transformations to analyze and enumerate the resulting combinatorial objects.
The results are new, correct and detailed. The paper is original and doesn't contradict to ethical or policy issues, the question posed by authors is new and well defined, the methods used by authors are appropriate and well described, the data are sound and well controlled, the discussion and conclusions are well balanced, the title and abstract convey the obtained results, the writing is acceptable, the paper contains good scientific results.
I recommend the paper to indexing.

Is the work clearly and accurately presented and does it cite the current literature?
Yes

Is the study design appropriate and is the work technically sound?
Yes

Are sufficient details of methods and analysis provided to allow replication by others?
Yes

If applicable, is the statistical analysis and its interpretation appropriate?
Yes

Are all the source data underlying the results available to ensure full reproducibility?
Yes

Are the conclusions drawn adequately supported by the results?

Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Mathematical analysis, complex analysis, topological algebra

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard.

Reviewer Report 11 July 2026

https://doi.org/10.5256/f1000research.197917.r497661

© 2026 Aubad A. This is an open access peer review report distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

?

Ali A. Aubad
[1] University of Baghdad, Baghdad, Iraq
[2] University of Baghdad, Baghdad, Iraq

The researchers aim to present a new class defined through a representation on the James abacus via a nested chain, which is denoted as the Ω-nested abacus. Several results are obtained within the manuscript. Despite the results obtained by the researchers, the manuscript needs to be carefully revised in its entirety, according to the comments attached below:
1-*References 4 and 9 do not have the definitions they are cited to support. Use appropriate references.*
2- Check all the symbols that you use in the manuscript. For example, *in Definition 2, replace k with b.*
3- *Definitions 2, 3, and 4 are not clearly introduced. For example, Definition 3 does not specify the runner e , and the chain structures within the N.C.A. definition are unclear.*
4- *Figures 4 and 5 require further verification and additional detail.*
5- *The researchers relied on a single case study, namely the case where b=d is odd. The authors must extend their analysis to validate the generality of their findings.*
6- *The authors are obliged to provide a mathematical proof for Remark 10.*
7- *The proof of Lemma 11 is incomplete; it needs more details on how the final formula is obtained. The same holds for Theorem 16.*
8- *The last sentence of the proof of Lemma 13 needs to be: "then the number of even chains is (b-1)/4, remove everything before that.*
9- *In the proof of Theorem 18, the researchers state: 'Since the MSPT-transformation acts simultaneously on all even chains.' However, this claim requires further explanation.*

10- *The researchers suggest a new family called 'Ω-nested Abacus', however, the definition of a nested abacus already exists. What improvement has been made in the researchers' definition?*
11- *There is not enough information in the 'Generating function with respect to chains' section. The proof of the G.F. formula is requested, and more details on the ECO methods are needed as well.*
12- *The authors should thoroughly review the entire manuscript for language and scientific formatting.*

Is the work clearly and accurately presented and does it cite the current literature?
Partly

Is the study design appropriate and is the work technically sound?
Partly

Are sufficient details of methods and analysis provided to allow replication by others?
Partly

If applicable, is the statistical analysis and its interpretation appropriate?
Not applicable

Are all the source data underlying the results available to ensure full reproducibility?
No source data required

Are the conclusions drawn adequately supported by the results?
Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Pure Mathematics (Algebraic Graph Theory)

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard, however I have significant reservations, as outlined above.

Author Response 21 Jul 2026

Eman Mohommed

We sincerely thank the reviewer for the careful reading of our manuscript and for the constructive comments. All remarks have been carefully considered, and the manuscript has been revised accordingly. A detailed response to each comment is provided below.

Comment 1
References 4 and 9 do not contain the definitions cited by the authors.
Response
Thank you for pointing out this issue. The cited references have been carefully rechecked, and the corresponding citations have been corrected. In particular, the definitions related to the Nested Chain Abacus have been revised and now consistently refer to the appropriate source. The manuscript has also been updated to ensure that every definition is supported by the correct reference.

Comment 2
The symbols should be checked. In Definition 2, the notation k **should be replaced by** b**.**

Response
The notation has been revised throughout the manuscript to ensure consistency. In particular, the parameter b is now used uniformly to denote the number of columns of the Nested Chain Abacus, while d denotes the number of rows. The notation has been checked throughout the manuscript to avoid ambiguity.

Comment 3
Definitions 2, 3 and 4 require further clarification.
Response
Definitions 2, 3 and 4 have been substantially rewritten to improve clarity and mathematical precision. The structure of the Nested Chain Abacus is now explicitly defined in terms of rows, columns, chains, and beta positions. Additional explanations have been included to clarify the geometric interpretation and the role of each chain.

Comment 4
Figures 4 and 5 require further verification and additional detail.
Response
The captions of Figures 4 and 5 have been revised to provide additional structural information. Figure 4 now explicitly describes the corresponding Nested Chain Abacus together with the associated beta-number sequence. Figure 5 has been presented as an independent example illustrating the notion of a connected chain, thereby supporting Definition 6 and improving the interpretation of the subsequent transformations.

Comment 5
The researchers relied on a single case study, namely the case where b=db=db=d **is odd. The authors must extend their analysis to validate the generality of their findings.**
Response
The manuscript has been revised to clarify the scope of the proposed model. Rather than treating the case b=d= odd as a single example, this condition is now incorporated directly into the definition of the Ω-nested Abacus. A separate remark has also been added explaining that an odd number of rows and columns guarantees the existence of a unique central position, which is an essential structural requirement for the proposed class. Consequently, all subsequent lemmas and theorems are formulated for the entire family of Ω-nested Abaci satisfying this condition, rather than for a single illustrative example.

Comment 6
The authors are obliged to provide a mathematical proof for Remark 10.
Response
A complete mathematical justification has been added. The revised proof explicitly shows that each admissible position in a chain produces exactly one SSPT-transformation and that every position is transformed at most once. Consequently, the maximal number of SSPT-transformations is equal to the number of positions in the chain.

Comment 7
The proof of Lemma 11 is incomplete; it needs more details on how the final formula is obtained. The same holds for Theorem 16.

Response
The proof of Lemma 11 has been expanded by explicitly explaining the geometric growth of successive chains. The derivation now shows why the second chain contains exactly eight positions and how each additional chain increases the boundary by eight positions, leading to the formula
$|C_n| = 8(n-1)$
Similarly, the proof of Theorem 16 has been rewritten to provide the missing intermediate arguments and improve the logical flow.

Comment 8
The last sentence of the proof of Lemma 13 needs to be revised.
Response
The proof has been simplified according to the reviewer's suggestion. The unnecessary sentence has been removed, and the proof now concludes directly with the statement that the number of even chains is
,
which follows immediately from the chain indexing.

Comment 9
The statement "Since the MSPT-transformation acts simultaneously on all even chains" requires further explanation.
Response
Additional justification has been incorporated into the proof of Theorem 18. The revised proof now explains that the chains of the Nested Chain Abacus are pairwise disjoint; therefore, each SSPT acts locally within its own even chain and does not affect any other chain. Consequently, the admissible transformations in different even chains are independent, allowing the MSPT-transformation to be interpreted as the simultaneous application of SSPTs to all even chains. The final counting formula follows directly from the multiplication principle.

Comment 10
There is not enough information in the "Generating function with respect to chains" section. The proof of the generating function and more details on the ECO method are required.
Response
This section has been substantially expanded. Additional material describing the ECO methodology, the generating tree, and the succession rule has been included. The recurrence relation has been derived explicitly, followed by a detailed derivation of the generating polynomial using the binomial theorem. Furthermore, the derivation of the bivariate generating function has been expanded by including the omitted intermediate algebraic steps and the application of the finite geometric-series identity. These revisions provide a complete mathematical justification of the generating function while preserving the original enumeration results.

Final Remark
We are grateful to the reviewer for the valuable comments and suggestions. We believe that the revisions have significantly improved the mathematical rigor, clarity, and presentation

of the manuscript, and we hope that the revised version satisfactorily addresses all concerns raised during the review process.

Competing Interests: No competing interests were disclosed.

Reviewer Report 25 May 2026

https://doi.org/10.5256/f1000research.197917.r485075

© **2026 Hashim H.** This is an open access peer review report distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Hayder R. Hashim
[1] University of Kufa, Kufa, Iraq
[2] University of Kufa, Kufa, Iraq

The authors have revised the manuscript in accordance with my comments. My only remaining suggestion to the authors is to carefully polish the language. Otherwise, I recommend acceptance for indexing.

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Number Theory and cryptography

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard.

Version 1

Reviewer Report 05 March 2026

https://doi.org/10.5256/f1000research.190675.r462385

© **2026 Hashim H.** This is an open access peer review report distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Hayder R. Hashim
[1] University of Kufa, Kufa, Iraq
[2] University of Kufa, Kufa, Iraq
[3] University of Kufa, Kufa, Iraq

The manuscript proposes a new class of polyominoes defined by a graphical representation referred to as the James Abacus or Nested Chain Abacus and attempts to enumerate this class using combinatorial transformations and the ECO method.

Since the results in the paper are interesting in number theory, I suggest accepting the paper after revising it according to the following remarks as the paper would benefit from improvements in clarity, organization, and presentation.

1. Lemma 11 and Theorem 12 refer to external theorems without clearly stating assumptions or applicability.
2. Corollary 13 and Theorem 14 contain inconsistent notation such as mixing b,p,r,e,d and unclear counting logic.
3. Polish the language carefully.
4. Every lemma and theorem must include a complete, self-contained proof.
5. All variables must be clearly defined and used consistently.
6. If results rely on previous publications, the exact statements must be cited and restated.
7. Try to reconstruct in more details the ECO method application properly.
8. Redo or rewrite the derivation of the generating function theorem from first principles, with a more clear form.

Is the work clearly and accurately presented and does it cite the current literature?
Yes

Is the study design appropriate and is the work technically sound?
Partly

Are sufficient details of methods and analysis provided to allow replication by others?
No

If applicable, is the statistical analysis and its interpretation appropriate?
Not applicable

Are all the source data underlying the results available to ensure full reproducibility?
No source data required

Are the conclusions drawn adequately supported by the results?
Yes

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Number Theory

I confirm that I have read this submission and believe that I have an appropriate level of expertise to confirm that it is of an acceptable scientific standard, however I have significant reservations, as outlined above.

Author Response 19 Mar 2026

Eman Mohommed

We thank the reviewer for his comments
Comment 1:
Lemma 11 and theorem 12 refer to external theorems without clearly stating assumptions of applicability
Response: in the revised version, we have explicitly stated all required assumptions and clarified the applicability of the referenced result. In addition, we have added several lemmas and theorems related with lemma 11 and theorem 12 to ensure that lemma 11 and theorem 12 are fully self-contained and mathematically precise.
Comment 2:
Corollary 13 and Theorem 14 contain inconsistent notation, such as mixing b, p, r, e, d, and unclear counting logic
Response: we appreciate this observation. The notation has been carefully revised and unified throughout the paper. All variables are now clearly defined and used consistently.
Comment 3:
Polish the language carefully
Response: all manuscripts have been proofread and language has been improved.
Comment 4:
Every lemma and theorem must include a complete, self-contained proof.
Response: In the new version, all lemmas and theorems now include complete proof. Additional explanations have been added where necessary.
Comment 5:
All variables must be clearly defined and used consistently.
Response:
All variables have been systematically reviewed and defined at their first occurrence and then has been standardized across the paper to ensure consistency.
Comment 6:
If results rely on previous publications, the exact statements must be cited and restated
Response: we revised all relevant sections to explicitly cite the original source.
Comment 7,8:
Try to reconstruct in more detail the proper application of the ECO method.
Redo or rewrite the derivation of the generating function theorem from first.
Response: following the reviewer's suggestion, we have completely rewritten the derivation of the generating function.

Competing Interests: No competing interests were disclosed.

Reviewer Report 16 January 2026

https://doi.org/10.5256/f1000research.190675.r450050

© 2026 Qureshi A. This is an open access peer review report distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Ayesha Qureshi
[1] Sabanci Universitesi (Ringgold ID: 52991), Istanbul, Istanbul, Turkey
[2] Sabanci Universitesi (Ringgold ID: 52991), Istanbul, Istanbul, Turkey
[3] Sabanci Universitesi (Ringgold ID: 52991), Istanbul, Istanbul, Turkey

This paper introduces a class of polyominoes defined via a nested chain abacus representation and aims to enumerate this class using transformations and the ECO method. While the topic may be of interest, the manuscript in its current form lacks sufficient mathematical clarity and rigor.

Several definitions are unclear or inconsistent, and many results are stated without complete or verifiable proofs. In particular, the use of the ECO method and the derivation of the generating function require a more precise and rigorous formulation. Additionally, the presentation and language need substantial improvement. A major revision addressing these issues would be necessary before the conclusions can be considered supported by the results.

Is the work clearly and accurately presented and does it cite the current literature?
Partly

Is the study design appropriate and is the work technically sound?
Partly

Are sufficient details of methods and analysis provided to allow replication by others?
No

If applicable, is the statistical analysis and its interpretation appropriate?
Not applicable

Are all the source data underlying the results available to ensure full reproducibility?
No source data required

Are the conclusions drawn adequately supported by the results?
No

Competing Interests: No competing interests were disclosed.

Reviewer Expertise: Algebra

I confirm that I have read this submission and believe that I have an appropriate level of expertise to state that I do not consider it to be of an acceptable scientific standard, for reasons outlined above.

The benefits of publishing with F1000Research:

- Your article is published within days, with no editorial bias
- You can publish traditional articles, null/negative results, case reports, data notes and more
- The peer review process is transparent and collaborative
- Your article is indexed in PubMed after passing peer review
- Dedicated customer support at every stage

For pre-submission enquiries, contact research@f1000.com

F1000Research