STREAMING FLUID MOTION WITHIN A LATERALLY OSCILLATING SPHERE WITH DENSITY STRATIFICATION

by D. KONG

(Environmental Control Systems, The Boeing Company, Long Beach, CA 90808, USA)

S. ZHANG

(Convergent Science, Madison, WI 53719, USA)

A. PENKOVA

(Department of Aerospace & Mechanical Engineering, University of Southern California, Los Angeles, CA 90089-1453, USA)

A. Y. REDNIKOV

(TIPs Laboratory, Université libre de Bruxelles, Brussels 1050, Belgium)
and

S. S. SADHAL[†]

(Department of Aerospace & Mechanical Engineering, University of Southern California, Los Angeles, CA 90089-1453, USA and Department of Ophthalmology, University of Southern California, Los Angeles, CA 90033-4682, USA)

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Summary

In this investigation, we have examined the fundamental problem of streaming motion in a liquid-filled sphere undergoing lateral oscillations. Such motion can be externally created, or exists in spacecrafts where g-jitter is well known. The important point here is that for spatially constant liquid density, such an internal problem is degenerate, when no non-trivial oscillatory flow and hence no streaming occur. However, this totally changes in the presence of some density stratification, which may be due to compositional and/or thermal non-uniformities. To clarify the phenomenon in simplest possible terms, we here just consider a constant volumetric heating source within the liquid and isothermal container walls. Proceeding with oscillatory displacement of the spherical shell, we assume a high-frequency limit relative to the viscous and thermal times, and a small displacement amplitude relative to the container (sphere) size. Treating the oscillations as a perturbation to an otherwise stationary shell, two steady streaming contributions are revealed. One is driven in the bulk of the liquid, which is atypical in the incompressible-liquid limit. The other classically originates in the Stokes layer at the boundary also engaging the bulk by viscosity. Even if the former is asymptotically greater here, it does not turn out to be practical to outright disregard the latter. The reason

[†]Corresponding author <sadhal@usc.edu>

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is the particularly low prefactor values arising in the former, which is typical for the internal problem. The streaming pattern consists of two or four axially symmetric vortices, which are dependent on the result of the competition between the two contributions.

Keywords: oscillatory flow; streaming; density stratification.

1. Introduction

The principal idea is that oscillatory fluid motion interacting with a solid or fluid boundary can give rise to streaming motion which corresponds to nonzero time-averaged flow in fluid medium that is otherwise dominated by zero-mean oscillatory flow. This effect is largely confined to a thin viscous boundary layer (called the shear-wave layer or the Stokes layer) of thickness $\delta \simeq (\nu/\omega)^{1/2}$ on the surface. Here ν is the kinematic viscosity and ω is the frequency related to the oscillatory flow. While the medium outside the layer vibrates irrotationally (as in a sound field), the one within the layer is forced to vibrate rotationally (i.e. with vorticity) because its motion has to conform to the no-slip condition on wall. Conversely, a vibrating body in an otherwise quiescent fluid also leads to irrotational fluid motion within the Stokes layer at the boundary of the body and serves as the source of streaming. The fundamental analysis in this regard was carried out by Riley (**1**) who treated the streaming flow around a sphere of radius R oscillating laterally with frequency ω and velocity amplitude U_0 in an infinite fluid with kinematic viscosity ν. The analysis was conducted for $Re = U_0R/\nu \ll 1$ and $\varepsilon = U_0/(\omega R) \ll 1$. A frequency parameter $M^2 = i\omega R^2/\nu$ was defined and cases corresponding to $|M|^2 \gg 1$ and $|M|^2 \ll 1$ were treated. For $|M|^2 \gg 1$, the analysis predicted the existence of a recirculating zone within the Stokes layer. This layer also represents a significant dissipation mechanism, and provides a strong force in driving streaming. Studies on streaming go back well over a hundred years and this has been the subject of many investigations including the fundamental works of Lord Rayleigh (**2**), Nyborg (**3, 4**), Lane (**5**), Wang (**6**), Riley (**1**) and Kelly (**7**). Since several reviews in the area of streaming phenomena are available in the published literature (**8 to 10**), we shall not indulge into the background material.

Rotational oscillations also bring about streaming motion, and concerning such motion of a sphere, Gopinath (**11**) has provide a perturbation solution developed along the lines of Riley's work (**1**). Here again, recirculating vortices are shown to exist in the Stokes layer adjacent to the surface of the sphere. While Gopinath's work (**11**) pertains to the exterior of a sphere, the corresponding interior problem has been addressed by Bonfiglio *et al.* (**12**) and Repetto *et al.* (**13, 14**) who investigated the streaming motion of the liquid in a spherical shell resulting from its rotational oscillations as a model of saccadic motion of eyes. For the steady streaming case, Repetto, Siggers and Stochino (**14**) showed that streaming flow travels from the centre of the sphere towards the poles and then towards the equatorial plane to return to the centre. With regard to the interior problem, we note that transverse oscillations do not lead to a nontrivial flow field unless the fluid has a density stratification. In other words, for spatially uniform density in a laterally oscillating shell, there is no fluid motion within the shell relative to its surface, and the system behaves like a solid body.

The exterior problem with non-uniform density distribution around a sphere in an oscillatory environment was studied by Amin (**15**). This analytical investigation (**15**) was conducted for an isothermal sphere at rest relative to the environment (i.e. no oscillations relative to the

environment), but with a g-jitter acting on a medium. The temperature field outside the sphere with the far field at a lower value gives rise to a non-uniform density distribution which brings about an accelerative body force. As already mentioned, Amin's work (**15**) pertains to the exterior region around a sphere and for this case, the initial state of steady conduction exists. Our investigation however corresponds to the interior of the sphere for which the steady conduction problem has a trivial solution unless there are distributed heat sources within the interior. Furthermore, as stated already, transverse oscillations do not produce any interior fluid motion unless there is spatial density variation. This is the fundamental problem that we are investigating. We therefore consider the simplest possible case where the density variation is the result of uniformly distributed heat sources, with constant surface temperature. While thermal effects combined with oscillatory flow has been the subject of many investigations (**16**), the streaming phenomena resulting from thermally driven density stratification with oscillatory motion has received little attention and it appears that the only available published works are those of Merkin (**16**) and Amin (**15**).

We are presently addressing a basic analytically tractable problem in which the starting point is a steady state, and the spatial density variation is resulting from thermal stimulation. We state once again that rotational oscillations do result in streaming motion even for uniform density systems (**11** to **14**, **17**) whereas lateral oscillations of a fluid-filled shell do not produce streaming.

The problem is defined as follows. We consider a fluid-filled spherical shell of radius R in which the fluid is heated uniformly (e.g. by electrical means) while the shell surface is maintained at a uniform temperature. Upon reaching a steady paraboloidal temperature profile, the sphere starts undergoing externally driven transverse oscillations at frequency ω. While with uniform fluid density, no fluid motion is expected and as already mentioned, density variation accompanied by the accelerative force due to the oscillations brings about fluid motion. To our knowledge, the analysis of such motion for the interior of a shell has not been reported. For the purpose of isolating the effect of oscillations, we are not considering gravity in the current problem. We are applying a methodology here that is similar to Amin's (**15**) and follow the perturbation expansion for a set of two small parameters. As done by Riley (**1**) and also stated earlier, one is the usual Reynolds number Re based on the velocity amplitude of oscillatory flow in the sphere U_0 so that $Re = U_0 R/\nu$. The other is $\varepsilon = U_0/(\omega R)$ which is considered sufficiently small so that $\varepsilon/Re = \nu/\omega R^2 \ll 1$. Here $\varepsilon/Re = 1/|M|^2$ where $M^2 = i\omega R^2/\nu$ is Riley's (**1**) parameter defined earlier. The perturbation procedure consists of outer expansions in terms of Re and ε/Re, and inner scaling based on $(\varepsilon/Re)^{1/2}$.

2. Model formulation and theoretical development

We consider a spherical container of radius R, with its centre located at the origin. The temperature of this container surface is kept constant at $T = 0$. An incompressible Newtonian fluid of density ρ and kinematic viscosity ν fills the domain, and continuous volumetric heating is applied to the fluid. In an undisturbed state, the fluid has a steady paraboloidal temperature distribution. At that point, an external lateral oscillatory disturbance with frequency ω and amplitude a is initiated on the sphere, i.e. linear vibrations of the sphere are started in the direction along an axis. Such motion can be described by a displacement $z = a\sin(\omega t^*)$ of the shell surface, considering that the motion is back and forth in the z-direction [see e.g. (**1**, **6**, **18**)]. If we now shift the coordinate system to a non-inertial reference frame attached to the

sphere, the fluid within experiences an acceleration $a\omega^2\sin(\omega t^*)$. Since the fluid density is non-uniform, given by the linear relationship $\rho = \rho_0\beta T$ with β as the thermal expansion coefficient and ρ_0 as the fluid density at reference temperature corresponding to the value at the wall, the acceleration gives rise to an effective body force $\rho_0 T\omega^2 a\beta\sin(\omega t^*)$. As done by Amin (**15**), we are invoking the Boussinesq approximation that is often used in thermally-driven buoyant convection. The key assumption here is that density variation is small but it matters when it shows up as a body force (such as $\rho_0\beta Tg$) and happens to be the significant driving mechanism. In the current analysis, the acceleration is $a\omega^2\sin(\omega t^*)$ instead of g. This problem is three-dimensional axisymmetric, and we formulate it in spherical coordinates (r,θ,φ). The coordinate system origin coincides with the centre of the spherical container. We have depicted a schematic of the laterally oscillating sphere in Fig. 1. The dimensional continuity, Navier-Stokes and energy equations are

$$\begin{aligned} \nabla^*\cdot(\mathbf{u}^*) &= 0,\\ \frac{\partial \mathbf{u}^*}{\partial t^*} + u^*\cdot\nabla^*\mathbf{u}^* &= -\frac{1}{\rho}\nabla^* p^* + \nu\nabla^{*2}\mathbf{u}^* + T^*\omega^2 a\beta\sin(\omega t^*)\hat{\mathbf{k}},\\ \frac{\partial T^*}{\partial t^*} + \mathbf{u}^*\cdot\nabla^* T^* &= \alpha\left(\nabla^* T^* + \frac{q_0'''}{k}\right), \end{aligned} \tag{2.1}$$

with boundary conditions

$$\begin{aligned} \mathbf{u}^* = 0 \quad \text{and} \quad T^* = 0 \qquad & r^* = R,\\ u^* \text{ is finite} \qquad & r^* = 0, \end{aligned} \tag{2.2}$$

and the initial condition

$$T^* = T_c\left(1 - \frac{r^{*2}}{R^2}\right) \quad \text{at} \quad t^* = 0, \tag{2.3}$$

where $T_c = q_0'''R/(6k)$ is the centrepoint temperature relative to the wall temperature, α is the thermal diffusivity, k is the thermal conductivity, q_0''' is the volumetric heat generation rate, and $\hat{\mathbf{k}}$ is the unit vector in the direction of oscillation.

With the velocity, radial distance, time, pressure, temperature scaled with respect to U_0, R, ω^{-1}, $\rho\omega RU_0$ and T_c, the dimensionless variables are introduced as

$$u = \frac{\mathbf{u}^*}{U_0}, r = \frac{r^*}{R}, \nabla = R\nabla^*, t = \omega t^*, p = \frac{p^*}{\rho\omega RU_0}, T = \frac{T^*}{T_c}, \tag{2.4}$$

where $U_0 = \omega a\beta T_c$. The stream function, ψ, related to the velocity components is introduced as,

$$u_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, \quad u_\theta = -\frac{\left(1-\mu^2\right)^{-\frac{1}{2}}}{r}\frac{\partial\psi}{\partial r}, \tag{2.5}$$

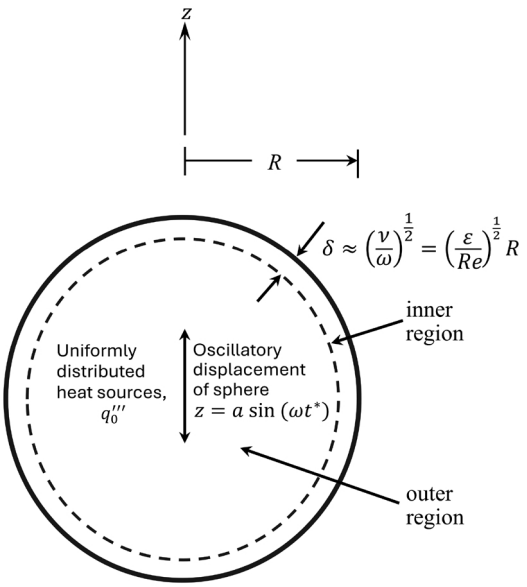


Fig. 1 A schematic of the laterally oscillating fluid-filled spherical shell of radius R. The shell displacement amplitude is a

where $\mu = \cos\theta$. The dimensionless equations take the forms of the vorticity and energy equations,

$$\begin{aligned} \frac{\partial}{\partial t}\left(D^2\psi\right) + \frac{\varepsilon}{r^2}\left[\frac{\partial\left(\psi, D^2\psi\right)}{\partial(r,\mu)} + 2D^2\psi L_1\psi\right] &= \frac{\varepsilon}{Re}D^4\psi + \left(1-\mu^2\right)L_2 T\sin t, \\ \frac{\partial T}{\partial t} + \frac{\varepsilon}{r^2}\left[\frac{\partial(\psi,T)}{\partial(r,\mu)}\right] &= \frac{\varepsilon}{RePr}\left(D^2T + \frac{2}{r^2}L_2T + 6\right), \end{aligned} \tag{2.6}$$

where

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial\mu^2}, \quad L_1 = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\mu}, \quad \text{and} \quad L_2 = r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial\mu} \tag{2.7}$$

are the differential operators. The parameters ε and Re have been defined in the Introduction but for the sake of clarity, we shall repeat the corresponding expressions here. Thus,

$$\varepsilon = \frac{U_0}{\omega R}, \qquad Re = \frac{U_0 R}{\nu}, \qquad \text{and} \qquad Pr = \frac{\nu}{\alpha}. \tag{2.8}$$

The associated boundary and initial conditions of the above equations are

$$\begin{aligned} \psi = \frac{\partial\psi}{\partial r} = 0 \quad \text{and} \quad T = 0 \quad \text{on} \quad r = 1, \\ \psi = O(r^2) \qquad \text{as} \quad r \to 0, \\ T = 1 - r^2 \qquad \text{at} \quad t = 0. \end{aligned} \tag{2.9}$$

Now we consider the solutions of the governing equations for $Re \ll 1$, $\mathrm{Pr} = O(1)$ and $(\varepsilon/Re)^{\frac{1}{2}} \ll 1$. As an inner boundary layer, the Stokes shear layer has a thickness of $O(\varepsilon/Re)^{\frac{1}{2}}$, or for that matter $O(1/|M|)$. Following Riley (**1**) as well as Amin (**15**), we construct the inner variables based on the Stokes layer thickness so that

$$\Psi = \left(\frac{Re}{2\varepsilon}\right)^{\frac{1}{2}}\psi, \quad \mathscr{T} = T \quad \text{and} \quad \eta = \left(\frac{Re}{2\varepsilon}\right)^{\frac{1}{2}}(1-r). \tag{2.10}$$

This allows us to begin a perturbation expansion in powers of $(\varepsilon/Re)^{\frac{1}{2}}$, i.e.

$$\begin{aligned} \psi &= \psi_0 + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\psi_1 + \left(\frac{\varepsilon}{Re}\right)\psi_2 + \cdots, \\ T &= T_0 + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}T_1 + \left(\frac{\varepsilon}{Re}\right)T_2 + \cdots. \end{aligned} \tag{2.11}$$

Considering that we also have $Re \ll 1$ as a small parameter in Equation (2.6), we proceed with expanding ψ_i and T_i, $i = 1,2,3,\cdots$, in powers of Re. Thus,

$$\begin{aligned} \psi &= (\psi_{00} + Re\psi_{01} + Re^2\psi_{02} + \cdots) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}(\psi_{10} + Re\psi_{11} + Re^2\psi_{12} + \cdots) \\ &\quad + \left(\frac{\varepsilon}{Re}\right)(\psi_{20} + Re\psi_{21} + Re^2\psi_{22} + \cdots) + \cdots, \\ T &= (T_{00} + ReT_{01} + Re^2T_{02} + \cdots) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}(T_{10} + ReT_{11} + Re^2T_{12} + \cdots) \\ &\quad + \left(\frac{\varepsilon}{Re}\right)(T_{20} + ReT_{21} + Re^2T_{22} + \cdots) + \cdots. \end{aligned} \tag{2.12}$$

The resulting equations for Ψ and $\mathscr{T}$ in the inner region are

$$\begin{aligned} &\frac{\partial}{\partial t}(\bar{D}^2\Psi) + \varepsilon\left[1 + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + 3\left(\frac{2\varepsilon}{Re}\right)\eta^2 + \cdots\right]\left[-\frac{\partial(\Psi,\bar{D}^2\Psi)}{\partial(\eta,\mu)} + 2\bar{D}^2\Psi\bar{L}_1\Psi\right] \\ &\quad = \frac{1}{2}\bar{D}^4\Psi + (1-\mu^2)\bar{L}_2\mathscr{T}\sin t, \\ &\frac{\partial\mathscr{T}}{\partial t} - \varepsilon\left[1 + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + 3\left(\frac{2\varepsilon}{Re}\right)\eta^2 + \cdots\right]\frac{\partial(\Psi,\mathscr{T})}{\partial(\eta,\mu)} \\ &\quad = \frac{1}{Pr}\left\{\frac{1}{2}\bar{D}^2\mathscr{T} + \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left[1 + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + 3\left(\frac{2\varepsilon}{Re}\right)\eta^2 + \cdots\right]\bar{L}_2\mathscr{T} + 6\left(\frac{\varepsilon}{Re}\right)\right\}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\bar{D}^2 &= \frac{\partial^2}{\partial\eta^2} + \left(\frac{2\varepsilon}{Re}\right)(1-\mu^2)\left[1 + 2\left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + 3\left(\frac{2\varepsilon}{Re}\right)\eta^2 + \cdots\right]\frac{\partial^2}{\partial\mu^2},\\
\bar{L}_1 &= -\frac{\mu}{1-\mu^2}\frac{\partial}{\partial\eta} + \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\left[1 + \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta + \left(\frac{2\varepsilon}{Re}\right)\eta^2 + \cdots\right]\frac{\partial}{\partial\mu},\\
\bar{L}_2 &= -\left[1 - \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\eta\right]\frac{\partial}{\partial\eta} - \left(\frac{2\varepsilon}{Re}\right)^{\frac{1}{2}}\mu\frac{\partial}{\partial\mu}.
\end{aligned} \tag{2.14}$$

We now carry out the expansions for Ψ and $\mathscr{T}$ in the same manner as Equation (2.12),

$$\begin{aligned}
\Psi = &\left(\Psi_{00} + Re\Psi_{01} + Re^2\Psi_{02} + \cdots\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(\Psi_{10} + Re\Psi_{11} + Re^2\Psi_{12} + \cdots\right)\\
&+ \left(\frac{\varepsilon}{Re}\right)\left(\Psi_{20} + Re\Psi_{21} + Re^2\Psi_{22} + \cdots\right) + \cdots,
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
\mathscr{T} = &\left(\mathscr{T}_{00} + Re\,\mathscr{T}_{01} + Re^2\,\mathscr{T}_{02} + \cdots\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(\mathscr{T}_{10} + Re\,\mathscr{T}_{11} + Re^2\,\mathscr{T}_{12} + \cdots\right)\\
&+ \left(\frac{\varepsilon}{Re}\right)\left(\mathscr{T}_{20} + Re\,\mathscr{T}_{21} + Re^2\,\mathscr{T}_{22} + \cdots\right) + \cdots.
\end{aligned} \tag{2.16}$$

Substitution of Equation (2.12) into momentum Equations (2.6) and (2.13) results in the equations for the leading-order as

$$\begin{aligned}
\frac{\partial}{\partial t}\left(D^2\psi_{00}\right) &= (1-\mu^2)L_2T_{00}\sin t,\\
\frac{\partial T_{00}}{\partial t} &= 0,\\
\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{00}}{\partial\eta^2}\right) - \frac{1}{2}\frac{\partial^4\Psi_{00}}{\partial\eta^4} &= -(1-\mu^2)\frac{\mathscr{T}_{00}}{\partial\eta}\sin t,\\
\frac{\partial\mathscr{T}_{00}}{\partial t} - \frac{1}{2Pr}\frac{\partial^2\mathscr{T}_{00}}{\partial\eta^2} &= 0,
\end{aligned} \tag{2.17}$$

with boundary conditions and initial condition

$$\begin{aligned}
\Psi_{00} = \frac{\partial\Psi_{00}}{\partial\eta} = 0 \quad \text{and} \quad \mathscr{T}_{00} = 0 \quad &\text{on} \quad \eta = 0,\\
\psi_{00} = O(r^2) \qquad\qquad &\text{as} \quad r \to 0,\\
T_{00} = 1 - r^2 \qquad\qquad &\text{at} \quad t = 0.
\end{aligned} \tag{2.18}$$

After some algebra we obtain the solutions to order $O(1)$ in the outer region as

$$\begin{aligned}\psi_{00} &= \psi_{00}^{(u)} = \left(B_{00}r^2 + \frac{1}{10}r^4\right)(1-\mu^2)e^{it} + \text{c.c.},\\ T_{00} &= T_{00}^{(s)} = 1 - r^2,\end{aligned} \tag{2.19}$$

where the thermal field to this order is entirely conductive since the convective transport is higher order.

For the inner region we have

$$\begin{aligned}\Psi_{00} = \Psi_{00}^{(u)} &= b_{00}\left\{\eta + \frac{1}{2}(1-i)\left[e^{-(1+i)\eta} - 1\right]\right\}(1-\mu^2)e^{it} + \frac{1}{4}a_{00}\eta^2(1-\mu^2)e^{it}\\ &\quad + \text{c.c.},\\ \mathscr{T}_{00} = \mathscr{T}_{00}^{(s)} &= a_{00}(\mu)\eta,\end{aligned} \tag{2.20}$$

where c.c. denotes the complex conjugate, and following Riley's (**1**) notation, the superscripts (u) and (s) refer to unsteady and steady parts, respectively. The coefficients in the solutions of each region can be obtained by matching with the solutions in the other region, according to the requirements

$$\Psi|_{\eta\to\infty} = \left(\frac{Re}{2\varepsilon}\right)^{\frac{1}{2}}\psi|_{r\to 1}, \tag{2.21}$$

and

$$\mathscr{T}|_{\eta\to\infty} = T|_{r\to 1}. \tag{2.22}$$

After matching, we obtain

$$B_{00} = -\frac{1}{10}, \quad b_{00} = -\frac{1}{5}, \quad a_{00} = 0 \tag{2.23}$$

at which point all the coefficients are determined. Thus, the leading-order solutions are found to be as follows. For the outer region,

$$\begin{aligned}\psi_{00} &= \psi_{00}^{(u)} = \left(-\frac{1}{10}r^2 + \frac{1}{10}r^4\right)(1-\mu^2)e^{it} + \text{c.c.},\\ T_{00} &= T_{00}^{(s)} = 1 - r^2.\end{aligned} \tag{2.24}$$

For the inner region around the container wall,

$$\begin{aligned}\Psi_{00} = \Psi_{00}^{(u)} &= -\frac{1}{5}\left\{\eta + \frac{1}{2}(1-i)\left[e^{-(1+i)\eta} - 1\right]\right\}(1-\mu^2)e^{it} + \text{c.c.},\\ \mathscr{T}_{00} = \mathscr{T}_{00}^{(s)} &= 0.\end{aligned} \tag{2.25}$$

To order $O(Re)$, the flow is still studied separately in the inner and outer regions. The stream function and temperature satisfy

$$\begin{aligned}
\frac{\partial}{\partial t}\left(D^2\psi_{01}\right) &= \left(1-\mu^2\right)L_2T_{01}\sin t,\\
\frac{\partial T_{01}}{\partial t} &= 0,\\
\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{01}}{\partial\eta^2}\right) - \frac{1}{2}\frac{\partial^4\Psi_{01}}{\partial\eta^4} &= -\left(1-\mu^2\right)\frac{\mathscr{T}_{01}}{\partial\eta}\sin t,\\
\frac{\partial\mathscr{T}_{01}}{\partial t} - \frac{1}{2Pr}\frac{\partial^2\mathscr{T}_{01}}{\partial\eta^2} &= 0,
\end{aligned} \tag{2.26}$$

with boundary conditions and initial condition

$$\begin{aligned}
\Psi_{01} = \frac{\partial\Psi_{01}}{\partial\eta} = 0 \quad &\text{and} \quad \mathscr{T}_{01} = 0 \qquad \text{on} \quad \eta = 0,\\
\psi_{01} = O(r^2) \quad &\text{and} \quad T_{01} = O(1) \quad \text{as} \quad r \to 0,\\
T_{01} = 0 \qquad\qquad &\qquad\qquad\quad \text{at} \quad t = 0.
\end{aligned} \tag{2.27}$$

The equation for $\mathscr{T}_{01}$ has the solution

$$\mathscr{T}_{01}^{(s)} = a_{01}(\mu)\eta, \tag{2.28}$$

but as we see in Equation (2.26) for T_{01}, it is evident that $T_{01} = T_{01}^{(s)}$ only. To determine T_{01}, the equation for T_{21} given by $O(\varepsilon)$ terms of Equation (2.6) is used, which is

$$\frac{\partial T_{21}}{\partial t} + \frac{1}{r^2}\left[\frac{\partial(\psi_{00},T_{00})}{\partial(r,\mu)}\right] = \frac{1}{Pr}\left(D^2T_{01} + \frac{2}{r^2}L_2T_{01}\right). \tag{2.29}$$

The steady part of Equation (2.29) yields the equation for T_{01} as

$$D^2T_{01}^{(s)} + \frac{2}{r^2}L_2T_{01}^{(s)} = \nabla^2T_{01}^{(s)} = 0, \tag{2.30}$$

where L_2 is defined in Equation (2.7). Equation (2.30) has the standard solution in terms of Legendre polynomials,

$$T_{01} = T_{01}^{(s)} = \sum_{n=0}^{\infty} A_{01n}r^nP_n(\mu). \tag{2.31}$$

All the coefficients are determined by the matching requirement provided in Equation (2.22), and thus the solutions are

$$\begin{aligned}
T_{01} &= T_{01}^{(s)} = 0\\
\mathscr{T}_{01} &= \mathscr{T}_{01}^{(s)} = 0.
\end{aligned} \tag{2.32}$$

The equations for $\psi_{01}^{(u)}$ and $\Psi_{01}^{(u)}$ become

$$\begin{aligned} D^2\psi_{01}^{(u)} &= 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{01}^{(u)}}{\partial\eta^2}\right) &= \frac{1}{2}\frac{\partial^4\Psi_{01}^{(u)}}{\partial\eta^4}. \end{aligned} \tag{2.33}$$

With the homogeneous boundary conditions, these equations have solutions

$$\psi_{01}^{(u)} = \Psi_{01}^{(u)} = 0. \tag{2.34}$$

The steady part of equation for Ψ_{01} satisfies

$$\frac{\partial^4\Psi_{01}^{(s)}}{\partial\eta^4} = 0, \tag{2.35}$$

By satisfying the boundary conditions given by Equation (2.27), the solution takes the form

$$\Psi_{01}^{(s)} = b_{01}(\mu)\eta^3 + c_{01}(\mu)\eta^2. \tag{2.36}$$

For the outer region, the $O(\varepsilon)$ terms in Equation (2.6) give

$$\begin{aligned} -\frac{\partial}{\partial t}\left(D^2\psi_{21}\right) + D^4\psi_{01} &= -\left(1-\mu^2\right)L_2T_{21}\sin t + \frac{1}{r^2}\frac{\partial\left(\psi_{00}^{(u)}, D^2\psi_{00}^{(u)}\right)}{\partial(r,\mu)}, \\ &+ \frac{2}{r^2}D^2\psi_{00}^{(u)}L_1\psi_{00}^{(u)}, \end{aligned} \tag{2.37}$$

Since $\psi_{00} \propto \cos t$, we can conclude that the right hand side of Equation (2.37) contains both time-independent terms and time-dependent terms with $\cos 2t$ behaviour.

The unsteady part of Equation (2.29)

$$\frac{\partial T_{21}^{(u)}}{\partial t} = -\frac{1}{r^2}\left[\frac{\partial\left(\psi_{00}^{(u)}, T_{00}^{(s)}\right)}{\partial(r,\mu)}\right] \tag{2.38}$$

has solution

$$T_{21}^{(u)} = -i\left(-\frac{2}{5}r + \frac{2}{5}r^3\right)\mu e^{it} + \text{c .c.}. \tag{2.39}$$

Upon substitution of $T_{21}^{(u)}$ into Equation (2.37), the steady part of the equation for $\psi_{01}^{(s)}$ emerges as

$$D^4\psi_{01}^{(s)} = -\frac{4}{5}r^3\mu\left(1-\mu^2\right). \tag{2.40}$$

After the boundary conditions are applied, the solution is obtained as

$$\psi_{01}^{(s)} = \left(-\frac{1}{630}r^7 + B_{011}r^5 + B_{012}r^3\right)\mu(1-\mu^2). \tag{2.41}$$

Matching yields the solutions

$$\begin{aligned} \psi_{01}^{(s)} &= \left(-\frac{1}{630}r^7 + \frac{1}{315}r^5 - \frac{1}{630}r^3\right)\mu(1-\mu^2) = -\frac{1}{630}r^3(1-r^2)^2\mu(1-\mu^2), \\ \Psi_{01}^{(s)} &= 0. \end{aligned} \tag{2.42}$$

Here $\psi_{01}^{(s)}$ corresponds to $O(Re)$ in Equation (2.12). However with the small value of the coefficient, $\psi_{01}^{(s)}$ has a maximum magnitude

$$|\psi_{01}^{(s)}|_{max} = \frac{32\sqrt{7}}{1512630} = 5.60 \times 10^{-5}. \tag{2.43}$$

This has implications on how far we can tolerate pushing Re values upwards.

The equations for the stream function and the temperature to order $O(Re^2)$ are

$$\begin{aligned} \frac{\partial}{\partial t}(D^2\psi_{02}) &= (1-\mu^2)L_2T_{02}\sin t, \\ \frac{\partial T_{02}}{\partial t} &= 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{02}}{\partial\eta^2}\right) - \frac{1}{2}\frac{\partial^4\Psi_{02}}{\partial\eta^4} &= -(1-\mu^2)\frac{\mathscr{T}_{02}}{\partial\eta}\sin t, \\ \frac{\partial\mathscr{T}_{02}}{\partial t} - \frac{1}{2Pr}\frac{\partial^2\mathscr{T}_{02}}{\partial\eta^2} &= 0. \end{aligned} \tag{2.44}$$

The boundary and initial conditions are

$$\begin{aligned} \Psi_{02} = \frac{\partial\Psi_{02}}{\partial\eta} = 0 \quad \text{and} \quad \mathscr{T}_{02} = 0 \qquad &\text{on} \quad \eta = 0, \\ \psi_{02} = O(r^2) \quad \text{and} \quad T_{02} = O(1) \quad &\text{as} \quad r \to 0, \\ T_{02} = 0 \qquad\qquad\qquad\qquad &\text{at} \quad t = 0. \end{aligned} \tag{2.45}$$

The above equations for T_{02} and $\mathscr{T}_{02}$ imply that

$$\begin{aligned} T_{02} &= T_{02}^{(s)}, \\ \mathscr{T}_{02}^{(s)} &= a_{02}(\mu)\eta. \end{aligned} \tag{2.46}$$

As for T_{01}, the equations to order $O(\varepsilon Re)$ are used to solve for T_{02} and ψ_{02}, which are

$$\frac{\partial T_{22}}{\partial t}+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{01}^{(s)},T_{00}^{(s)}\right)}{\partial(r,\mu)}\right]+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{00}^{(u)},T_{01}\right)}{\partial(r,\mu)}\right]=\frac{1}{Pr}\left(D^2T_{02}^{(s)}+\frac{2}{r^2}L_2T_{02}^{(s)}\right), \tag{2.47}$$

$$\begin{aligned}-\frac{\partial}{\partial t}\left(D^2\psi_{22}\right)+D^4\psi_{02}&=-(1-\mu^2)L_2T_{22}\sin t+\frac{1}{r^2}\frac{\partial\left(\psi_{00}^{(u)},D^2\psi_{01}^{(s)}\right)}{\partial(r,\mu)}\\&+\frac{1}{r^2}\frac{\partial\left(\psi_{01}^{(s)},D^2\psi_{00}^{(u)}\right)}{\partial(r,\mu)}+\frac{2}{r^2}D^2\psi_{00}^{(u)}L_1\psi_{01}^{(s)}+\frac{2}{r^2}D^2\psi_{01}^{(s)}L_1\psi_{00}^{(u)}.\end{aligned} \tag{2.48}$$

In the Equation (2.48) above for stream function, T_{22} in the term $-(1-\mu^2)L_2T_{22}\sin t$ is shown to be zero in Equation (2.72). This is used later to obtain ψ_{02}.

The steady part of Equation (2.47) above for the temperature is

$$\begin{aligned}\frac{1}{Pr}\left(D^2T_{02}^{(s)}+\frac{2}{r^2}L_2T_{02}^{(s)}\right)&=\frac{1}{r^2}\left[\frac{\partial\left(\psi_{01}^{(s)},T_{00}^{(s)}\right)}{\partial(r,\mu)}\right]\\&=\left(-\frac{1}{315}r^6+\frac{2}{315}r^4-\frac{1}{315}r^2\right)(1-3\mu^2),\end{aligned} \tag{2.49}$$

which has the solution

$$T_{02}^{(s)}=\left[A_{02}r^2+\frac{Pr}{630}\left(-\frac{1}{33}r^8+\frac{1}{9}r^6-\frac{1}{7}r^4\right)\right](1-3\mu^2). \tag{2.50}$$

Using the matching requirement Equation (2.22), we obtain

$$\begin{aligned}T_{02}&=T_{02}^{(s)}=\frac{Pr}{630}\left(-\frac{1}{33}r^8+\frac{1}{9}r^6-\frac{1}{7}r^4+\frac{43}{693}r^2\right)(1-3\mu^2)\\\mathscr{T}_{02}&=\mathscr{T}_{02}^{(s)}=0.\end{aligned} \tag{2.51}$$

The equations to order $O\left\{(\varepsilon/Re)^{\frac{1}{2}}\right\}$ take the form,

$$\begin{aligned}\frac{\partial}{\partial t}\left(D^2\psi_{10}\right)&=(1-\mu^2)L_2T_{10}\sin t,\\\frac{\partial T_{10}}{\partial t}&=0,\\\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{10}}{\partial\eta^2}\right)-\frac{1}{2}\frac{\partial^4\Psi_{10}}{\partial\eta^4}&=(1-\mu^2)\left(-\frac{\mathscr{T}_{01}}{\partial\eta}+\sqrt{2}\eta\frac{\mathscr{T}_{00}}{\partial\eta}-\sqrt{2}\mu\frac{\mathscr{T}_{00}}{\partial\mu}\right)\sin t,\\\frac{\partial\mathscr{T}_{10}}{\partial t}&=\frac{1}{Pr}\left(\frac{1}{2}\frac{\partial^2\mathscr{T}_{10}}{\partial\eta^2}-\sqrt{2}\frac{\partial\mathscr{T}_{00}}{\partial\eta}\right),\end{aligned} \tag{2.52}$$

and the corresponding boundary and initial conditions are

$$\begin{aligned} \Psi_{10} = \frac{\partial \Psi_{10}}{\partial \eta} = 0 \quad \text{and} \quad \mathscr{T}_{10} = 0 \qquad \text{on} \quad \eta = 0, \\ \psi_{10} = O(r^2) \quad \text{and} \quad T_{10} = O(1) \quad \text{as} \quad r \to 0, \\ T_{10} = 0 \qquad\qquad\qquad \text{at} \quad t = 0. \end{aligned} \tag{2.53}$$

The appropriate form of temperature solutions T_{10} and $\mathscr{T}_{10}$ should be

$$\begin{aligned} T_{10} &= T_{10}^{(s)}, \\ \mathscr{T}_{10}^{(s)} &= a_{10}(\mu)\eta. \end{aligned} \tag{2.54}$$

Again, we look for the equations to order $O\left\{(\varepsilon/Re)^{\frac{3}{2}}\right\}$, which are

$$\frac{\partial T_{30}}{\partial t} = \frac{1}{Pr}\left(D^2 T_{10}^{(s)} + \frac{2}{r^2} L_2 T_{10}^{(s)}\right), \tag{2.55}$$

$$-\frac{\partial}{\partial t}\left(D^2 \psi_{30}\right) + D^4 \psi_{10} = -\left(1-\mu^2\right) L_2 T_{30} \sin t. \tag{2.56}$$

Upon solving the steady part of Equation (2.55) for the temperature,

$$D^2 T_{10}^{(s)} + \frac{2}{r^2} L_2 T_{10}^{(s)} = \nabla^2 T_{10}^{(s)} = 0, \tag{2.57}$$

the expression for $T_{10}^{(s)}$ is found to be

$$T_{10} = T_{10}^{(s)} = \sum_{n=0}^{\infty} A_{10n} r^n P_n(\mu). \tag{2.58}$$

The matching requirement Equation (2.22) leads to the solutions

$$\begin{aligned} T_{10} &= T_{10}^{(s)} = 0 \\ \mathscr{T}_{10} &= \mathscr{T}_{10}^{(s)} = 2\sqrt{2}\eta. \end{aligned} \tag{2.59}$$

Then the equations for steam functions in the inner and outer regions reduce to

$$\begin{aligned} \frac{\partial}{\partial t}\left(D^2 \psi_{10}\right) = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial^2 \Psi_{10}}{\partial \eta^2}\right) - \frac{1}{2}\frac{\partial^4 \Psi_{10}}{\partial \eta^4} = \left(-2\sqrt{2}\right)\left(1-\mu^2\right)\sin t. \end{aligned} \tag{2.60}$$

Equation (2.55) for the temperature together with Equation (2.58) implies

$$\begin{aligned} \frac{\partial T_{30}^{(u)}}{\partial t} &= 0, \\ T_{30}^{(u)} &= 0, \end{aligned} \tag{2.61}$$

and therefore the right-hand side of Equation (2.56) for stream function $-(1-\mu^2)L_2 T_{30}\sin t$ contains no steady part and is proportional to $\sin t$. The steady parts of Equation (2.56) and Equation (2.60) form the equations for the stream function $\psi_{10}^{(s)}$ and $\Psi_{10}^{(s)}$,

$$\begin{aligned} D^4\psi_{10}^{(s)} &= 0, \\ \frac{\partial^4\Psi_{10}^{(s)}}{\partial\eta^4} &= 0. \end{aligned} \tag{2.62}$$

The general solutions can be expressed as

$$\begin{aligned} \psi_{10}^{(s)} &= \left(B_{101}r^5 + B_{102}r^3\right)\mu\left(1-\mu^2\right), \\ \Psi_{10}^{(s)} &= b_{10}(\mu)\eta^3 + c_{10}(\mu)\eta^2. \end{aligned} \tag{2.63}$$

Upon satisfying the matching requirement Equation (2.21), the solutions are found to be

$$\psi_{10}^{(s)} = \Psi_{10}^{(s)} = 0. \tag{2.64}$$

The unsteady components of Equation (2.60) give the equations for $\psi_{10}^{(u)}$ and $\Psi_{10}^{(u)}$ as

$$\begin{aligned} D^2\psi_{10}^{(u)} &= 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{10}^{(u)}}{\partial\eta^2}\right) - \frac{1}{2}\frac{\partial^4\Psi_{10}^{(u)}}{\partial\eta^4} &= \left(-2\sqrt{2}\right)\left(1-\mu^2\right)\sin t, \end{aligned} \tag{2.65}$$

which have the general solutions

$$\begin{aligned} \psi_{10}^{(u)} &= C_{10}r^2\left(1-\mu^2\right)e^{it} + \text{c.c.}, \\ \Psi_{10}^{(u)} &= d_{10}\left\{\eta + \frac{1}{2}(1-i)\left[e^{-(1+i)\eta} - 1\right]\right\}\left(1-\mu^2\right)e^{it} + \frac{\sqrt{2}}{2}\eta^2\left(1-\mu^2\right)e^{it} + \text{c .c.}. \end{aligned} \tag{2.66}$$

According to the matching requirement given by Equation (2.21), the solutions are found as

$$\begin{aligned} \psi_{10}^{(u)} &= \frac{\sqrt{2}}{10}(1-i)r^2\left(1-\mu^2\right)e^{it} + \text{c.c.}, \\ \Psi_{10}^{(u)} &= \frac{\sqrt{2}}{10}\left\{5\eta^2 - 2(1-i)\eta + 2i\left[e^{-(1+i)\eta} - 1\right]\right\}\left(1-\mu^2\right)e^{it} + \text{c.c.}. \end{aligned} \tag{2.67}$$

Following a similar procedure as for obtaining the solutions at the orders $O(1)$, $O(Re)$, $O(Re^2)$ and $O\left\{(\varepsilon/Re)^{\frac{1}{2}}\right\}$, we obtain the solutions at the orders $O\left\{(\varepsilon Re)^{\frac{1}{2}}\right\}$, $O\left\{(\varepsilon/Re)^{\frac{1}{2}}Re^2\right\}$, $O(\varepsilon/Re)$, $O(\varepsilon)$, $O(\varepsilon Re)$, $O(Re^2)$ and $O(Re^3)$ as summarized below.

$$\left.\begin{aligned}
&\psi_{11} = \psi_{11}^{(u)} = \psi_{11}^{(s)} = \Psi_{11}^{(u)} = 0,\\
&\Psi_{11} = \Psi_{11}^{(s)} = -\frac{2\sqrt{2}}{315}\eta^2\mu(1-\mu^2),\\
&T_{11} = T_{11}^{(u)} = T_{11}^{(s)} = \mathscr{T}_{11} = \mathscr{T}_{11}^{(u)} = \mathscr{T}_{11}^{(s)} = 0;
\end{aligned}\right\} \tag{2.68}$$

$$\left.\begin{aligned}
&\psi_{12}^{(s)} = \Psi_{12}^{(s)} = 0,\\
&\psi_{12} = \psi_{12}^{(u)} = \frac{2\sqrt{2}Pr}{1576575}(1-i)r^4(1-\mu^2)(1-5\mu^2)e^{it} + \text{c.c.},\\
&\Psi_{12} = \Psi_{12}^{(u)} = \frac{4\sqrt{2}Pr}{1091475}\eta^2(1-\mu^2)e^{it} + \frac{Pr}{630}\left\{\frac{12\sqrt{2}}{3465}\eta^2 - \frac{16\sqrt{2}}{5005}(1-i)\eta\right.\\
&\qquad\qquad \left. + \frac{16\sqrt{2}i}{5005}\left[e^{-(1+i)\eta}-1\right]\right\}(1-\mu^2)(1-5\mu^2)e^{it} + \text{c.c.},\\
&T_{12} = T_{12}^{(u)} = T_{12}^{(s)} = \mathscr{T}_{12}^{(u)} = 0,\\
&\mathscr{T}_{12} = \mathscr{T}_{12}^{(s)} = \frac{8\sqrt{2}Pr}{218295}\eta(1-3\mu^2);
\end{aligned}\right\} \tag{2.69}$$

$$\left.\begin{aligned}
&\psi_{20}^{(s)} = \Psi_{20}^{(s)} = 0,\\
&\psi_{20} = \psi_{20}^{(u)} = -\frac{2}{5}ir^2(1-\mu^2)e^{it} + \text{c.c.},\\
&\Psi_{20} = \Psi_{20}^{(u)} = \frac{1}{5}\left\{-4\eta^3 + (1-i)\eta^2 + 4i\eta + 2(1+i)\left[e^{-(1+i)\eta}-1\right]\right.\\
&\qquad\qquad \left. - i\eta e^{-(1+i)\eta}\right\}(1-\mu^2)e^{it} + \text{c.c.},\\
&T_{20} = T_{20}^{(u)} = T_{20}^{(s)} = \mathscr{T}_{20}^{(u)} = 0,\\
&\mathscr{T}_{20} = \mathscr{T}_{20}^{(s)} = -2\eta^2;
\end{aligned}\right\} \tag{2.70}$$

$$\left.\begin{aligned}
\psi_{21}^{(s)} &= -\frac{1}{10}\left(r^5 - r^3\right)\mu\left(1-\mu^2\right),\\
\Psi_{21}^{(s)} &= \left[\frac{16}{315}\eta^3 + \frac{1}{5}\eta - \frac{21}{50} + \frac{1}{50}e^{-2\eta} + \frac{6}{25}e^{-\eta}\sin\eta + \frac{2}{5}e^{-\eta}\cos\eta\right.\\
&\quad \left. + \frac{4}{25}\eta e^{-\eta}\sin\eta\right]\mu\left(1-\mu^2\right),\\
\psi_{21}^{(u)} &= \frac{i}{70}\left(r^5 - r^3\right)\mu\left(1-\mu^2\right)e^{2it} + \text{c.c.},\\
\Psi_{21}^{(u)} &= \left\{-\frac{i}{35}\eta - \frac{\sqrt{2}}{140}(1+i)\left[e^{-\sqrt{2}(1+i)\eta} - 1\right] + \frac{1}{25}(1+i)e^{-(1+i)\eta}\right.\\
&\quad \left. - \frac{2i}{25}\eta e^{-(1+i)\eta} + \frac{1}{200}(1+i)e^{-2(1+i)\eta}\right\}\mu\left(1-\mu^2\right)e^{2it} + \text{c.c.},\\
T_{21}^{(s)} &= \mathscr{T}_{21} = \mathscr{T}_{21}^{(u)} = \mathscr{T}_{21}^{(s)} = 0,\\
T_{21}^{(u)} &= -\frac{2i}{5}\left(r^3 - r\right)\mu e^{it} + \text{c.c.};
\end{aligned}\right\} \tag{2.71}$$

$$\left.\begin{aligned}
T_{22} = T_{22}^{(u)} = T_{22}^{(s)} = \mathscr{T}_{22}^{(u)} = 0,\\
\mathscr{T}_{22} = \mathscr{T}_{22}^{(s)} = \frac{16Pr}{218295}\eta^2\left(1-3\mu^2\right);
\end{aligned}\right\} \tag{2.72}$$

$$\left.\begin{aligned}
\psi_{02}^{(s)} &= \Psi_{02}^{(s)} = 0,\\
\psi_{02} = \psi_{02}^{(u)} &= \frac{Pr}{1260}\left[\left(\frac{1}{660}r^{10} - \frac{1}{135}r^8 + \frac{1}{70}r^6 - \frac{43}{3465}r^4 + \frac{167}{41580}r^2\right)\left(1-\mu^2\right)\right.\\
&\quad \left. + \left(\frac{1}{715}r^{10} - \frac{1}{165}r^8 + \frac{1}{105}r^6 - \frac{73}{15015}r^4\right)\left(1-\mu^2\right)\left(1-5\mu^2\right)\right]e^{it} + \text{c.c.},\\
\Psi_{02} = \Psi_{02}^{(u)} &= \frac{4Pr}{1576575}\left\{\eta + \frac{1}{2}(1-i)\left[e^{-(1+i)\eta} - 1\right]\right\}\left(1-\mu^2\right)\left(1-5\mu^2\right)e^{it} + \text{c.c.};
\end{aligned}\right\} \tag{2.73}$$

As noted for $\psi_{01}^{(s)}$, the expression $\psi_{02}^{(s)}$ corresponding to $O(Re^2)$ is also very small and we may be able to push Re to values greater than 0.1 (perhaps up to 0.5) and still have meaningful results.

$$\left.\begin{aligned}
&\psi_{03}^{(u)} = \Psi_{03} = \Psi_{03}^{(u)} = \Psi_{03}^{(s)} = 0,\\
&\psi_{03}^{(s)} = \left\{\frac{1}{16488}\left[\frac{Pr}{630}\left(\frac{64}{1155}\right)+\frac{34}{77175}\right]r^{13} - \frac{1}{7476}\left[\frac{Pr}{630}\left(\frac{8}{45}\right)+\frac{226}{231525}\right]r^{11}\right.\\
&\quad + \frac{1}{2760}\left[\frac{Pr}{630}\left(\frac{48}{245}\right)+\frac{94}{138915}\right]r^{9} - \frac{1}{708}\left[\frac{Pr}{630}\left(\frac{3688}{45045}\right)+\frac{2}{14175}\right]r^{7}\\
&\quad + \left[\frac{Pr}{630}\left(\frac{12641817026}{130810084730325}\right)+\frac{41739197}{806814648476550}\right]r^{5}\\
&\quad \left.+ \left[\frac{Pr}{630}\left(-\frac{458817116}{14534453858925}\right)+\frac{5026003}{806814648476550}\right]r^{3}\right\}\mu(1-\mu^2)\\
&\quad + \left\{-\frac{1}{14940}\left[\frac{Pr}{630}\left(\frac{276}{5005}\right)+\frac{2}{77175}\right]r^{13}\right.\\
&\quad + \frac{1}{6460}\left[\frac{Pr}{630}\left(\frac{8}{55}\right)+\frac{44}{694575}\right]r^{11} - \frac{1}{2164}\left[\frac{Pr}{630}\left(\frac{76}{735}\right)+\frac{26}{694575}\right]r^{9}\\
&\quad + \left[\frac{Pr}{630}\left(\frac{9881128}{230970928825}\right)+\frac{67808}{5596603275375}\right]r^{7}\\
&\quad \left.- \left[\frac{Pr}{630}\left(\frac{105362602}{7622040651225}\right)+\frac{145541}{50369429478375}\right]r^{5}\right\}(1-\mu^2)(3\mu-7\mu^3),\\
&T_{03} = T_{03}^{(u)} = T_{03}^{(s)} = \mathscr{T}_{03} = \mathscr{T}_{03}^{(u)} = \mathscr{T}_{03}^{(s)} = 0.
\end{aligned}\right\} \tag{2.74}$$

We rewrite here the expansions (2.12), (2.15) and (2.16), but omitting the zero terms, i.e.

$$\begin{aligned}
\psi = &\left(\psi_{00}^{(u)} + Re\psi_{01}^{(s)} + Re^2\psi_{02}^{(u)} + Re^3\psi_{03}^{(s)} + \cdots\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(\psi_{10}^{(u)} + Re^2\psi_{12}^{(u)} + \cdots\right)\\
&+ \left(\frac{\varepsilon}{Re}\right)\left[\psi_{20}^{(u)} + Re\left(\psi_{21}^{(u)} + \psi_{21}^{(s)}\right) + \cdots\right] + \cdots,
\end{aligned} \tag{2.75}$$

$$\begin{aligned}
T = &\left(T_{00}^{(s)} + Re^2T_{02}^{(s)} + O(Re^4)\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(O(Re^3)\right)\\
&+ \left(\frac{\varepsilon}{Re}\right)\left(ReT_{21}^{(u)} + O(Re^3)\right) + \cdots,
\end{aligned} \tag{2.76}$$

$$\begin{aligned}
\Psi = &\left(\Psi_{00}^{(u)} + Re^2\Psi_{02}^{(u)} + O(Re^4)\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(\Psi_{10}^{(u)} + Re\Psi_{11}^{(s)} + Re^2\Psi_{12}^{(u)} + \cdots\right)\\
&+ \left(\frac{\varepsilon}{Re}\right)\left[\Psi_{20}^{(u)} + Re\left(\Psi_{21}^{(u)} + \Psi_{21}^{(s)}\right) + \cdots\right] + \cdots,
\end{aligned} \tag{2.77}$$

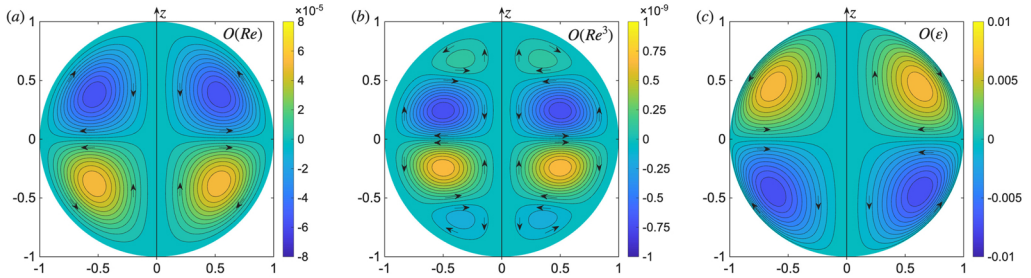


Fig. 2 The streaming flow pattern on r-θ plane with $Pr = 1$. (a) $\psi_{01}^{(s)}$, (b) $\psi_{03}^{(s)}$, (c) $\psi_{21}^{(s)}$. The colour bars refer to the streamline numerical values

and

$$\begin{aligned}\mathscr{T} = \left(O(Re^4)\right) + \left(\frac{\varepsilon}{Re}\right)^{\frac{1}{2}}\left(\mathscr{T}_{10}^{(s)} + Re^2\,\mathscr{T}_{12}^{(s)} + \cdots\right) \\ + \left(\frac{\varepsilon}{Re}\right)\left(\mathscr{T}_{20}^{(s)} + Re^2\,\mathscr{T}_{22}^{(s)} + \cdots\right) + \cdots.\end{aligned} \tag{2.78}$$

3. Results

As we can see from Equations (2.75) to (2.78), the thermal response to the oscillatory disturbance manifests itself at $O(Re^2)$. The same is true for the inner stream function. The inner temperature field $\mathscr{T}$ exhibits the same phenomenon at even higher order. In Fig. 2, the steady streaming patterns on the r-θ plane at different orders $O(Re)$, $O(Re^3)$ and $O(\varepsilon)$ are shown. The arrows present the steady component of velocity at each order. For all cases the oscillations are in the z-direction. Based on the expressions of streaming solutions provided in Equations (2.42), (2.71) and (2.74), four streaming vortices are generated at both $O(Re)$ and $O(\varepsilon)$. But the streaming flow patterns at the two orders are quite different. For $\psi_{01}^{(s)}$, the streaming vortex in the first quadrant on the $\varphi = 0$ plane is in the counter-clockwise direction, while it is clockwise for $\psi_{21}^{(s)}$. The centres of streaming vortices for $\psi_{01}^{(s)}$ are closer to the centre of spherical container. The values of the stream function $\psi_{21}^{(s)}$ are much higher in magnitude than those of $\psi_{01}^{(s)}$. This has implications for $\psi^{(s)}$ discussed in the next paragraph. At $O(Re^3)$, the streaming flow pattern has four axially symmetric vortices. Since the intensity of steady streaming at order $O(Re^3)$ is not comparable to that at lower order $O(Re)$, the stream function $\psi_{03}^{(s)}$ does not make a contribution in the properties of streaming vortices as great as the stream function $\psi_{01}^{(s)}$.

Steady streaming patterns for different ε values are plotted in Fig. 3. For small ε, the stream function $\psi_{01}^{(s)}$ plays a dominant role (Fig. 3a). As the ε value increases, the vortex changes direction from counter-clockwise to clockwise in the first quadrant on the $\varphi = 0$ plane (Fig. 3c). At the value of ε around 0.001, the transition occurs, as shown in Fig. 3b. The flow directions of vortices indicate that, when transition occurs, the stream function $\psi_{01}^{(s)}$ dominates the region close to the centre of container, while the stream function $\psi_{21}^{(s)}$ has a greater influence in the

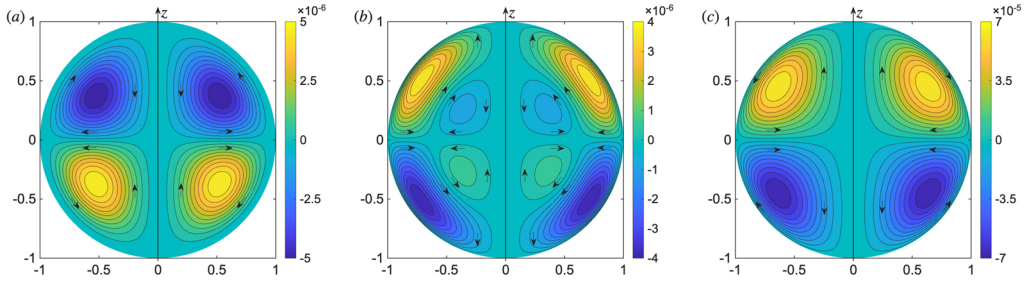


Fig. 3 The streamlines of $\psi^{(s)} = Re\psi_{01}^{(s)} + Re^3\psi_{03}^{(s)} + \varepsilon\psi_{21}^{(s)}$ on r-θ plane with $Re = 0.1$ and $Pr = 1$. (a) $\varepsilon = 0.0001$, (b) $\varepsilon = 0.001$, (c) $\varepsilon = 0.01$. The colour bars refer to the streamline numerical values

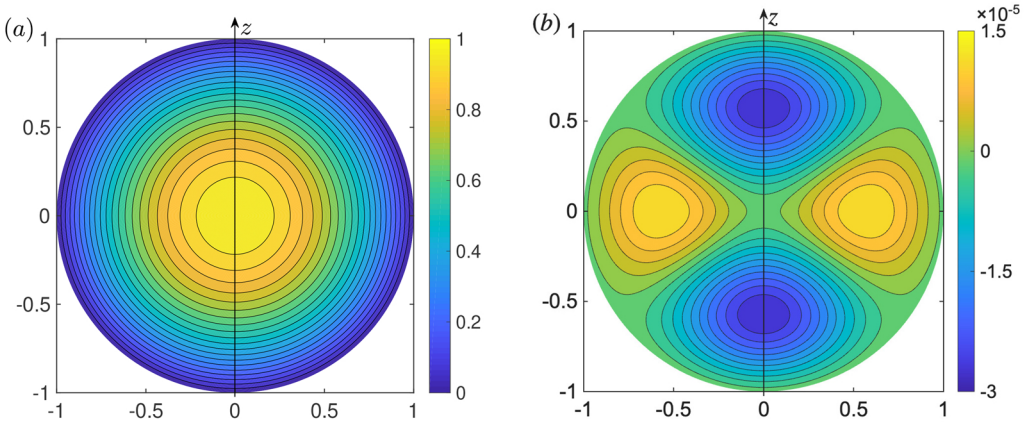


Fig. 4 The steady part of composite temperature profile on r-θ plane with $Pr = 1$. (a) $T_{00}^{(s)}$, (b) $T_{02}^{(s)}$. The colour bars refer to the temperature numerical values

region close to container boundary. Careful examination of the magnitudes of $\psi_{01}^{(s)}$, $\psi_{03}^{(s)}$ and $\psi_{21}^{(s)}$ shows that $\psi_{21}^{(s)}$ is considerably larger than $\psi_{01}^{(s)}$, owing to the relative values of the coefficients. Therefore at $Re = 0.1$ and $\varepsilon = 0.01$, $\varepsilon\psi_{21}^{(s)}$ has a larger and opposite contribution in comparison to $Re\psi_{01}^{(s)}$, thus explaining the reversal in the orientation of the streamlines.

Figure 4 shows the steady composite temperature profiles on the r-θ plane. The leading order of steady temperature $T_{00}^{(s)}$ results directly from the initial temperature distribution. The $T_{02}^{(s)}$ values is much smaller than the $T_{00}^{(s)}$ values. The maximum $T_{02}^{(s)}$ values are located on a circle of radius around 0.6 on the $\theta = \pi/2$ plane.

4. Discussion

While the lateral oscillatory motion of a fluid-filled spherical container produces no relative motion when the density is uniform, any density stratification of the fluid brings about a net accelerative body force leading to fluid motion as well as streaming. For the present case, we considered a physically realizable situation in which the thermal source within the fluid provides the temperature distribution accompanied by density stratification. With an initial steady state paraboloidal temperature profile, the oscillatory motion leads to a streaming pattern that consists of two or four axially symmetric vortices depending on the values of Re and ε. The ε contribution is originating from the Stokes layer, similar to the cases of uniform density, when such a contribution is the only one. In contrast, the Re contribution is attributable to an effective steady force in the bulk of the liquid, which results from the primary oscillatory flow being rotational in a stratified environment. On the contrary, for uniform density, the primary oscillatory flow is irrotational in the bulk, hence no bulk steady forcing is generated. Effectively, this results in no relative motion between the sphere and the fluid contained therein. For the parameter values considered ($Re = 0.1$ and $0.01 \leqslant \varepsilon \leqslant 0.0001$), the effect of oscillations on the temperature profile is quite weak.

On the question of the ranges of Re and ε, we need to bear in mind that $(\varepsilon/Re)^{\frac{1}{2}}$ also needs to be small. If we were to require $(\varepsilon/Re)^{\frac{1}{2}} < 0.1$, then $\varepsilon < 0.01Re$ putting severe limits on ε when Re were to remain < 0.1. However, keeping in mind the values of $O(Re)$ and $O(Re^2)$ terms [see discussions following Equations (2.42) and (2.73) on $\psi_{01}^{(s)}$ and $\psi_{02}^{(u)}$], we can push the limits of Re higher than 0.1 and still obtain meaningful results as already mentioned. Thus, it is not too far out of line to have $\varepsilon \approx 0.01$.

The thermal perturbation produced exhibits toroidal isotherms with the highest temperature elevation on a ring in the equatorial plane, approximately at a distance of $R/\sqrt{3}$ from the polar axis. The more challenging problem at hand is the fully time-dependent case of a solute introduced into the container with the same type of lateral oscillations. This is currently under investigation.

5. Conclusion

The basic streaming problem of an oscillating sphere has been the subject of several investigations for many decades (**1**, **6**, **11**, **19 to 21**). These works have been concerned with streaming in the exterior region of the sphere. The interior problem with streaming however has only received recent attention (**13**, **14**, **22**), but has been limited to rotational oscillations because for uniform density, there is no motion produced. Our investigation targeted thermally generated variable density with only lateral oscillations, and confirmed the existence of fluid motion as well as streaming, which have been quantified by the singular perturbation method.

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Conflict of Interest

The authors report no conflict of interest.

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APPENDIX A

In this Appendix, we show the derivation of Equations (2.73) and (2.74)

At order $O(Re^3)$, the equations are

$$\begin{aligned}
\frac{\partial}{\partial t}\left(D^2\psi_{03}\right) &= \left(1-\mu^2\right)L_2T_{03}\sin t,\\
\frac{\partial T_{03}}{\partial t} &= 0,\\
\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{03}}{\partial\eta^2}\right)-\frac{1}{2}\frac{\partial^4\Psi_{03}}{\partial\eta^4} &= -\left(1-\mu^2\right)\frac{\mathscr{T}_{03}}{\partial\eta}\sin t,\\
\frac{\partial\mathscr{T}_{03}}{\partial t}-\frac{1}{2Pr}\frac{\partial^2\mathscr{T}_{03}}{\partial\eta^2} &= 0,
\end{aligned} \tag{A.1}$$

with boundary and initial conditions

$$\begin{aligned}
\Psi_{03}=\frac{\partial\Psi_{03}}{\partial\eta}=0 \quad \text{and} \quad \mathscr{T}_{03}=0 \qquad &\text{on} \quad \eta=0,\\
\psi_{03}=O(r^2) \quad \text{and} \quad T_{03}=O(1) \quad &\text{as} \quad r\to 0,\\
T_{03}=0 \qquad\qquad\qquad\qquad &\text{at} \quad t=0.
\end{aligned} \tag{A.2}$$

It is reasonable to write the temperature solutions T_{03} and $\mathscr{T}_{03}$ as

$$\begin{aligned}
T_{03} &= T_{03}^{(s)},\\
\mathscr{T}_{03}^{(s)} &= a_{03}(\mu)\eta.
\end{aligned} \tag{A.3}$$

Again, Equation (A.1) does not provide enough information to solve for T_{03}. We need to consider the order $O(\varepsilon Re^2)$ terms in Equation (2.6) for the temperature, and following through gives

$$\begin{aligned}
&\frac{\partial T_{23}}{\partial t}+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{00}^{(u)},T_{02}^{(s)}\right)}{\partial(r,\mu)}\right]+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{02},T_{00}^{(s)}\right)}{\partial(r,\mu)}\right]+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{01}^{(s)},T_{01}\right)}{\partial(r,\mu)}\right]\\
&\quad=\frac{1}{Pr}\left(D^2T_{03}^{(s)}+\frac{2}{r^2}L_2T_{03}^{(s)}\right),
\end{aligned} \tag{A.4}$$

In the Equation (A.4) above, the stream function ψ_{02} is yet unknown, and needs to be solved first. Now we are able to write the steady part of Equation (2.48) for the stream function ψ_{02} as

$$D^4\psi_{02}^{(s)}=0, \tag{A.5}$$

which gives

$$\psi_{02}^{(s)}=\left(B_{021}r^5+B_{022}r^3\right)\mu\left(1-\mu^2\right). \tag{A.6}$$

The steady part of Equation (2.44) for the stream function in the inner region becomes

$$\frac{\partial^4 \Psi_{02}^{(s)}}{\partial \eta^4} = 0, \tag{A.7}$$

which has a solution of the form

$$\Psi_{02}^{(s)} = b_{021}(\mu)\eta^3 + b_{022}(\mu)\eta^2. \tag{A.8}$$

Satisfying the matching requirement Equation (2.21) gives

$$\psi_{02}^{(s)} = \Psi_{02}^{(s)} = 0. \tag{A.9}$$

Then the steady part of Equation (A.4) can be simplified to

$$D^2 T_{03}^{(s)} + \frac{2}{r^2} L_2 T_{03}^{(s)} = \nabla^2 T_{03}^{(s)} = 0. \tag{A.10}$$

The general solution takes the form

$$T_{03}^{(s)} = \sum_{n=0}^{\infty} A_{03n} r^n P_n(\mu). \tag{A.11}$$

Matching this solution in the outer region with the one in the inner region as in Equation (A.3), according to the requirement Equation (2.22), determined the coefficients and leads to

$$T_{03} = T_{03}^{(s)} = T_{03}^{(u)} = \mathscr{T}_{03} = \mathscr{T}_{03}^{(s)} = \mathscr{T}_{03}^{(u)} = 0. \tag{A.12}$$

By substituting Equation (A.12) into Equation (A.1), we now have the equation for stream functions in the inner and outer regions as

$$\begin{aligned} \frac{\partial}{\partial t}\left(D^2 \psi_{03}\right) &= 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial^2 \Psi_{03}}{\partial \eta^2}\right) - \frac{1}{2}\frac{\partial^4 \Psi_{03}}{\partial \eta^4} &= 0. \end{aligned} \tag{A.13}$$

The homogeneous boundary conditions (A2) imply

$$\psi_{03}^{(u)} = \Psi_{03}^{(u)} = 0. \tag{A.14}$$

The steady part of Equation (A.13) for $\Psi_{03}^{(s)}$

$$\frac{\partial^4 \Psi_{03}^{(s)}}{\partial \eta^4} = 0 \tag{A.15}$$

has the solution

$$\Psi_{03}^{(s)} = b_{03}(\mu)\eta^3 + c_{03}(\mu)\eta^2. \tag{A.16}$$

The equation for the stream function to order $O(\varepsilon Re^2)$ is used to solve for $\psi_{03}^{(s)}$, which is

$$\begin{aligned}-\frac{\partial}{\partial t}\left(D^2\psi_{23}\right)+D^4\psi_{03} &= -(1-\mu^2)L_2T_{23}\sin t+\frac{1}{r^2}\frac{\partial\left(\psi_{00}^{(u)},D^2\psi_{02}^{(u)}\right)}{\partial(r,\mu)}\\ &+\frac{1}{r^2}\frac{\partial\left(\psi_{02}^{(u)},D^2\psi_{00}^{(u)}\right)}{\partial(r,\mu)}+\frac{1}{r^2}\frac{\partial\left(\psi_{01}^{(s)},D^2\psi_{01}^{(s)}\right)}{\partial(r,\mu)}\\ &+\frac{2}{r^2}D^2\psi_{00}^{(u)}L_1\psi_{02}^{(u)}+\frac{2}{r^2}D^2\psi_{02}^{(u)}L_1\psi_{00}^{(u)}+\frac{2}{r^2}D^2\psi_{01}^{(s)}L_1\psi_{01}^{(s)}.\end{aligned}\tag{A.17}$$

The forcing function $-(1-\mu^2)L_2T_{23}^{(u)}\sin t$ is determined by solving the unsteady part of Equation (A.4), which is

$$\frac{\partial T_{23}^{(u)}}{\partial t}+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{00}^{(u)},T_{02}^{(s)}\right)}{\partial(r,\mu)}\right]+\frac{1}{r^2}\left[\frac{\partial\left(\psi_{02}^{(u)},T_{00}^{(s)}\right)}{\partial(r,\mu)}\right]=0.\tag{A.18}$$

In the above Equation (A.18), we need to get $\psi_{02}^{(u)}$ first before solving for $T_{23}^{(u)}$. The equation of stream function to order $O(Re^2)$, which is used to determine $\psi_{02}^{(u)}$, is

$$\frac{\partial}{\partial t}\left(D^2\psi_{02}^{(u)}\right)=(1-\mu^2)L_2T_{02}\sin t.\tag{A.19}$$

Upon substitution of Equation (2.51) into Equation (A.19) and integration with respect to t, the equation for $\psi_{02}^{(u)}$ becomes

$$\begin{aligned}D^2\psi_{02}^{(u)} &= \frac{Pr}{630}\left[\left(\frac{2}{15}r^8-\frac{2}{5}r^6+\frac{2}{5}r^4-\frac{86}{693}r^2\right)(1-\mu^2)\right.\\ &\left.+\left(\frac{6}{55}r^8-\frac{4}{15}r^6+\frac{6}{35}r^4\right)(1-\mu^2)(1-5\mu^2)\right]\cos t,\end{aligned}\tag{A.20}$$

which has the general solution

$$\begin{aligned}\psi_{02}^{(u)} &= \left[\frac{1}{2}B_{023}r^2+\frac{Pr}{1260}\left(\frac{1}{660}r^{10}-\frac{1}{135}r^8+\frac{1}{70}r^6-\frac{43}{3465}r^4\right)\right](1-\mu^2)e^{it}\\ &+\frac{1}{2}B_{024}r^3\mu(1-\mu^2)e^{it}\\ &+\left[\frac{1}{2}B_{025}r^4+\frac{Pr}{1260}\left(\frac{1}{715}r^{10}-\frac{1}{165}r^8+\frac{1}{105}r^6\right)\right](1-\mu^2)(1-5\mu^2)e^{it}+\text{c.c.}.\end{aligned}\tag{A.21}$$

The steady part of Equation (2.44) for $\Psi_{02}^{(u)}$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_{02}^{(u)}}{\partial\eta^2}\right)-\frac{1}{2}\frac{\partial^4\Psi_{02}^{(u)}}{\partial\eta^4}=0\tag{A.22}$$

has the solution of the form

$$\begin{aligned}\Psi_{02}^{(u)} &= b_{023}\left\{\eta+\frac{1}{2}(1-i)\left[e^{-(1+i)\eta}-1\right]\right\}(1-\mu^2)e^{it}\\ &+ b_{024}\left\{\eta+\frac{1}{2}(1-i)\left[e^{-(1+i)\eta}-1\right]\right\}\mu(1-\mu^2)e^{it}\\ &+ b_{025}\left\{\eta+\frac{1}{2}(1-i)\left[e^{-(1+i)\eta}-1\right]\right\}(1-\mu^2)(1-5\mu^2)e^{it}+\text{c.c.}.\end{aligned} \tag{A.23}$$

The coefficients in Equations (A.21) and (A.23) are determined by matching requirement Equation (2.21) and thus the solutions are

$$\begin{aligned}\psi_{02}^{(u)} &= \frac{Pr}{1260}\left(\frac{1}{660}r^{10}-\frac{1}{135}r^8+\frac{1}{70}r^6-\frac{43}{3465}r^4+\frac{167}{41580}r^2\right)(1-\mu^2)e^{it}\\ &+\frac{Pr}{1260}\left(\frac{1}{715}r^{10}-\frac{1}{165}r^8+\frac{1}{105}r^6-\frac{73}{15015}r^4\right)(1-\mu^2)(1-5\mu^2)e^{it}+\text{c.c.},\\ \Psi_{02}^{(u)} &= -\frac{8}{5005}\left(\frac{Pr}{630}\right)\left\{\eta+\frac{1}{2}(1-i)\left[e^{-(1+i)\eta}-1\right]\right\}(1-\mu^2)(1-5\mu^2)e^{it}+\text{c. c.}.\end{aligned} \tag{A.24}$$

Then the solution of Equation (A.18) is obtained as

$$\begin{aligned}T_{23}^{(u)} &= i\frac{Pr}{630}\left(-\frac{1}{330}r^9+\frac{2}{135}r^7-\frac{1}{35}r^5+\frac{86}{3465}r^3-\frac{167}{20790}r\right)\mu e^{it}\\ &+i\frac{Pr}{630}\left(\frac{4}{55}r^9-\frac{10}{33}r^7+\frac{10}{21}r^5-\frac{74}{231}r^3+\frac{86}{1155}r\right)\mu(1-\mu^2)e^{it}\\ &+i\frac{Pr}{630}\left(-\frac{8}{165}r^9+\frac{2}{11}r^7-\frac{26}{105}r^5+\frac{482}{3465}r^3-\frac{86}{3465}r\right)\mu(1-3\mu^2)e^{it}\\ &+i\frac{Pr}{630}\left(-\frac{4}{715}r^9+\frac{4}{165}r^7-\frac{4}{105}r^5+\frac{292}{15015}r^3\right)\mu(3-5\mu^2)e^{it}+\text{c.c.}.\end{aligned} \tag{A.25}$$

Thus, Equation (A.17) reduces to

$$\begin{aligned}D^4\psi_{03}^{(s)} &= \left\{\left[\frac{Pr}{630}\left(\frac{64}{1155}\right)+\frac{34}{77175}\right]r^9-\left[\frac{Pr}{630}\left(\frac{8}{45}\right)+\frac{226}{231525}\right]r^7\right.\\ &+\left[\frac{Pr}{630}\left(\frac{48}{245}\right)+\frac{94}{138915}\right]r^5-\left.\left[\frac{Pr}{630}\left(\frac{3688}{45045}\right)+\frac{2}{14175}\right]r^3\right\}\mu(1-\mu^2)\\ &+\left\{-\left[\frac{Pr}{630}\left(\frac{276}{5005}\right)+\frac{2}{77175}\right]r^9+\left[\frac{Pr}{630}\left(\frac{8}{55}\right)+\frac{44}{694575}\right]r^7\right.\\ &-\left.\left[\frac{Pr}{630}\left(\frac{76}{735}\right)+\frac{26}{694575}\right]r^5\right\}(1-\mu^2)(3\mu-7\mu^3),\end{aligned} \tag{A.26}$$

The general solution of Equation (A.26) has the form

$$
\begin{aligned}
\psi_{03}^{(s)} &= \psi_{03h}^{(s)} + \psi_{03p}^{(s)},\\
\psi_{03h}^{(s)} &= \left(B_{0311}r^4 + B_{0312}r^2\right)\left(1-\mu^2\right) + \left(B_{0321}r^5 + B_{0322}r^3\right)\mu\left(1-\mu^2\right)\\
&\quad + \left(B_{0331}r^6 + B_{0332}r^4\right)\left(1-\mu^2\right)\left(1-5\mu^2\right)\\
&\quad + \left(B_{0341}r^7 + B_{0342}r^5\right)\left(1-\mu^2\right)\left(3\mu-7\mu^3\right),\\
\psi_{03p}^{(s)} &= \left\{\frac{1}{16488}\left[\frac{Pr}{630}\left(\frac{64}{1155}\right)+\frac{34}{77175}\right]r^{13} - \frac{1}{7476}\left[\frac{Pr}{630}\left(\frac{8}{45}\right)+\frac{226}{231525}\right]r^{11}\right.\\
&\quad \left. + \frac{1}{2760}\left[\frac{Pr}{630}\left(\frac{48}{245}\right)+\frac{94}{138915}\right]r^{9} - \frac{1}{708}\left[\frac{Pr}{630}\left(\frac{3688}{45045}\right)+\frac{2}{14175}\right]r^{7}\right\}\mu\left(1-\mu^2\right)\\
&\quad + \left\{-\frac{1}{14940}\left[\frac{Pr}{630}\left(\frac{276}{5005}\right)+\frac{2}{77175}\right]r^{13} + \frac{1}{6460}\left[\frac{Pr}{630}\left(\frac{8}{55}\right)+\frac{44}{694575}\right]r^{11}\right.\\
&\quad \left. - \frac{1}{2164}\left[\frac{Pr}{630}\left(\frac{76}{735}\right)+\frac{26}{694575}\right]r^{9}\right\}\left(1-\mu^2\right)\left(3\mu-7\mu^3\right),
\end{aligned}
\tag{A.27}
$$

Matching $\psi_{03}^{(s)}$ in the outer region with $\Psi_{03}^{(s)}$ in the inner region as in Equation (A.16) leads to the solutions as provided in Equation (2.74).